Comprehensive Curriculum

Geometry

Cecil J. Picard
State Superintendent of Education
© April 2005

Louisiana Department of Education
Geometry
Table of Contents

Unit 1: Geometric Patterns and Puzzles ................................................................. 1
Unit 2: Reasoning and Proof ................................................................................. 13
Unit 3: Parallel and Perpendicular Relationships .............................................. 21
Unit 4: Triangles and Quadrilaterals ................................................................. 30
Unit 5: Similarity and Trigonometry ................................................................. 49
Unit 6: Area, Surface Area, Polyhedra, and Volume ........................................ 61
Unit 7: Circles and Spheres .............................................................................. 72
Unit 8: Transformations ..................................................................................... 81
Geometry
Unit 1: Geometric Patterns and Puzzles

Time Frame: Approximately three weeks

Unit Description

This unit introduces the use of inductive reasoning to extend a pattern and then find the rule for generating the $n$th term in a sequence. Additionally, counting techniques and mathematical modeling, including line of best fit, will be used to find solutions to real-life problems.

Student Understandings

Students apply inductive reasoning to identify terms of a sequence by generating a rule for the $n$th term. Students recognize linear versus non-linear sets of data and can justify their reasoning. They understand when to apply counting techniques to solve real-life problems.

Guiding Questions

1. Can students give examples of correct and incorrect usage of inductive reasoning?
2. Can students use counting experiences to develop patterns for number of diagonals and sums of angles in polygons?
3. Can students state the characteristics of a linear set of data?
4. Can students determine the formula for finding the $n$th term in a linear data set?
5. Can students solve a real-life sequence problem based on counting?

Unit 1 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>Write the equation of a line of best fit for a set of 2-variable real-life data presented in table or scatter plot form, with or without technology (A-2-H) (D-2-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>17.</td>
<td>Compare and contrast inductive and deductive reasoning approaches to justify conjectures and solve problems (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
</tbody>
</table>
Sample Activities

Activity 1: Inductive Reasoning (GLE: 17)

The purpose of this activity is to provide students with the definition of inductive reasoning and to have them recognize when inductive reasoning is used in real-life situations. Provide the definition of inductive reasoning and give an example of inductive reasoning that students may encounter on a day-to-day basis (e.g., the mailman came to my house every day at noon for five days in a row. I deduce that the mailman will come today at 12 P.M.). Discuss the fact that one counter-example is sufficient to disprove a conjecture made when using the inductive reasoning process (e.g., the mailman came today at 3 P.M.). Ask students to give other real-life examples. Provide students with a variety of scenarios in which students can make a conjecture using inductive reasoning. Have students identify situations in which inductive reasoning might be used inappropriately (e.g., matters of coincidence rather than a true pattern).

Activity 2: Using Inductive Reasoning in Number and Picture Patterns (GLE: 17)

The purpose of this activity is to allow students to use inductive reasoning to find the next number or picture in the sequence. Additionally, students will indicate verbally or in writing the process for generating the next item. Provide ample practice exercises in each of these strategies, starting with fairly simple problems and progressing to more challenging problems. Examples are

- 1, 4, 9, 16, 25, 36, ____, _____. \textit{Solution: 49, 64 (The numbers are perfect squares: }1^2, 2^2, 3^2, 4^2, 5^2, 6^2 \ldots \text{)}
- 1, 3, 7, 15, 31, 63, ____, _____. \textit{Solution: 127, 255 (The differences between each set of two terms are 2, 4, 8, 16, 32, etc. The differences are doubled. The}
next difference should be 64, so 63 + 64 = 127. The next difference would be 128, so 128 + 127 = 255.)

- Draw the next picture in the pattern. Solution: A circle with an inscribed pentagon. The points on the circles increase by one in each picture, which are connected to make polygons.

![Image of geometric patterns](image_url)

Activity 3: Recognizing Linear Relationships in Table Formats (GLEs: 5, 20, 22, 25, 26, 27)

The purpose of this activity is to develop the strategy of looking for common differences between values to determine if a relationship is linear. This strategy will be used in future activities to generate the rule for finding the \( n \)th term in relationships that are linear.

Have students work in groups to generate terms in a sequence using a given rule or function.

Examples:

<table>
<thead>
<tr>
<th>Term</th>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>( n - 3 )</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After finding missing values, groups should determine that the solution is 3 less than the term and recognize that this is the relationship established by \( n \) and \( n - 3 \). Students should also recognize that the difference between the values is 1.

<table>
<thead>
<tr>
<th>Term</th>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>( 2n + 3 )</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After finding the values, groups should determine that the common difference between the values is 2.

After performing several such exercises, groups should determine that the common difference is the same as the coefficient for \( n \).
Provide students with a data set which is not linear for comparison. For example:

<table>
<thead>
<tr>
<th>Term</th>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>( n^2 )</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Discuss the differences between the data sets to determine what makes a data set linear or not linear.

The skills listed in the following activity will review concepts that were mastered in Algebra I. Have students work in pairs and provide each pair of students with a graphing calculator or access to a Microsoft Excel on a computer.

- Using the available technology, have students
  - plot the terms and values as ordered pairs for each of the examples, using the term numbers as the \( x \)-coordinates and the values as the \( y \)-coordinates (term, value).
  - generate the equation of the line of best fit.
  - recognize the relationship between the terms and values to be linear or not linear.
  - understand the relationship between the rule or function in the original problem and the equation of the line.
  - recognize the relationship between the common difference, the coefficient of \( n \), and the slopes of the lines.
  - determine the next two or three values using the common difference rather than the function or rule.
- Ask students to perform similar tasks using pencil and paper so that they may review manual methods of writing linear equations for a data set.

For additional practice, provide small groups of students with different data sets. Some data sets should be non-linear. During a reporting session, have groups explain how they determined whether or not their data set was linear. For linear data sets, students should give the equation of the line and indicate the steps used in determining the equation.

**Activity 4: Use a Formula to Find the \( n^{\text{th}} \) Term in a Pattern (GLEs: 5, 17, 20, 26, 27)**

This activity ties activities two and three together. Present a number pattern that is linear in nature to the class, but do not give them a table or formula. Using the techniques from the previous activities, ask students to generate the formula which describes the relationship of the linear data.

Examples are:
- 1, 3, 5, 7, 9, … Find the 20\textsuperscript{th} term. Solution: Students should realize that writing out terms through the 20\textsuperscript{th} will take a while. If they assign each term a number to represent \( n \) (1 for first term, 2 for second term, 3 for third term, etc.) they can then apply the technique of plotting points, generating the
equation for the line of best fit, then finding the 20th term. The formula is 2n – 1. The 20th term is 39.

- 4, 8, 12, 16, 20 … Find the 100th term. Solution: Formula 4n; 100th term 400
- 4, 9, 14, 19, 24, … Find the 67th term. Solution: Formula 5n-1; 67th term 334
- Students should also be required to develop the formulas without the use of technology.

Ask students to generate the nth term for picture patterns. Examples are:

- 

  How many sides will the 15th term have? Solution: n + 2; 17 sides
  Add two to the figure number, to determine the number of sides. For example, the 3rd figure has 5 sides.

- 

  What will the 23rd figure look like? Solution: Since the pattern repeats after four figures, students should realize that every term that is a multiple of four will look like the fourth figure. The nearest multiple to 23 is 20; the students should then continue the pattern—it is the 3rd figure.

Activity 5: Figurate Numbers (GLEs: 5, 20, 22, 26, 27)

In this activity, students will generate the formulas for finding the nth term in a square, rectangular, or triangular number patterns. Each of these is a non-linear number pattern.

Square Numbers

Begin by presenting the following diagram.

First, have students translate the picture pattern into a number pattern by counting the number of dots in each figure. The number pattern is 1, 4, 9, 16, 25…. Ask the students if the pattern is a linear one. They should tell you that the data cannot be linear since the difference between values is not constant. Some students may recognize immediately that the numbers are perfect squares, but many will not unless the teacher provides leading
questions for class discussion. If needed, ask students why the picture pattern is called a square number pattern. Lead students to recognize that the dots form squares and that the number of dots in each square is the same as the area of the square. It may be necessary to ask them what is meant by the term *perfect square* before students understand that the numbers in the number sequence are the squares of the counting numbers \( (1^2, 2^2, 3^2, 4^2 \ldots) \); therefore, the formula for generating the \( n \)th term is \( n^2 \). Have students recognize that it is important to know the characteristics of linear data sets (common difference between each two terms) in order to quickly identify those that are non-linear.

Have students enter the data from the picture pattern into their graphing calculators and create a scatter plot [i.e., \((1,1), (2,4), (3,9) (4,16) (5,25) (6,36)\)]. Students may need to make a chart like the one below in order to determine what ordered pairs to use.

<table>
<thead>
<tr>
<th>Figure</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Dots</td>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Determine that students understand that the scatter plot is the graphical representation of the non-linear data set in the same way that the graph of a line is the graphical representation of a linear data set. Guide students through the process for generating the regression equation for the data set. Help students make the connection between the regression equation, \( y = x^2 \), and the rule for generating the \( n \)th term in the square number pattern, \( n^2 \).

Rectangular Numbers

Provide students with the number pattern shown below.

![Rectangular Numbers](image)

Have students:
- Write the number pattern that is created when counting the dots in each figure. *Solution:* \((2, 6, 12, 20, 30)\)
- Determine if the number pattern is linear or non-linear by using the characteristics of linear data sets. Do not allow students to use the scatter plot feature on their calculators to determine this. Instead, have students indicate that the differences between each pair of numbers is not the same (i.e., the differences are \(4, 6, 8, 10\ldots\)); therefore, the data cannot be linear
- Use their graphing calculators to determine the regression equation once they have determined that the pattern is non-linear. *Solution:* \( y = x^2 \)
- Indicate how the equation relates to the number pattern and how the equation can be used to determine the number of dots for any figure in the picture.
pattern. Solution: If \( n \) represents a given figure, the number of dots for that figure is \( n^2 + n \) or \( n(n + 1) \). Each rectangle has a width the same as the figure number and a length which is one greater than the width; therefore the number of dots needed for any figure is the same as the area of the rectangle, \( n(n+1) \), where \( n \) is the width and the length is one more than the width.

Triangular Numbers

Provide students with the picture pattern below.

<table>
<thead>
<tr>
<th>Figure</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Dots (Triangular #)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Have students make a chart similar to the one below in which the student enters the number of dots needed to make each figure. Ask them to follow the procedure outlined above to determine whether the relationship is linear or non-linear and then to generate the regression equation.

Teacher Note: The number pattern is 1, 3, 6, 10, 15…. Some students may recognize that they can add 2 to the first value to get the second value, add 3 to the second value to get the third value, etc. They will want to say that the rule is to add the next whole number to the previous one. They need to understand that this indicates that the pattern is not linear, since the difference between values is not the same. Lead them to understand that this pattern cannot be the formula or rule for generating the \( n^{th} \) term since the pattern they see is based upon knowing a previous term.

The formula for generating the \( n^{th} \) term is \( \frac{n(n+1)}{2} \). The computer will show the regression equation as \( 0.5n^2 + 0.5n \). Notice that this is half of \( n(n + 1) \) which was the rule for the rectangular numbers. Show students that the triangular number pattern could also be drawn as

```
*   *   *   *
  ** ** ** **
    *** *** ***
      **** ****
```
Each of the patterns above is one-half of each of the rectangles below.

```plaintext
   **   ***   ****   *****
***   ****   *****
****   *****
*****
```

Therefore, if the area of the rectangle is \(n(n+1)\), the area of the triangle would be half as much.

Provide students with a variety of number patterns, some linear and some non-linear, with which to practice their skills. Activity 6 gives an example of some geometric situations in which these skills must be applied.

**Activity 6: Applying Patterns and Counting to Geometric Concepts (GLEs: 5, 17, 20, 22, 24, 26, 27)**

Have students engage in a discussion about the sum of angles in various polygons, recognizing that if all possible diagonals are drawn from one vertex, the sum of the angles in the resulting triangles is the same as the sum of the angles in the polygon. Have students identify a pattern and use the pattern to write the formula for finding the sum of the angles in an \(n\)-gon. This is a linear relationship. Ask students to determine the formula without a graphing calculator and allow them to verify their results using the calculator.

Students should also discuss the total number of diagonals which can be drawn in a polygon. Have students draw the diagonals in a triangle, quadrilateral, pentagon, hexagon, and heptagon. Ask students to identify a pattern and use the pattern to determine the number of diagonals in other polygons. They should recognize that it is not linear and explain how they know it is not linear. Have students create a graph using the data collected and generate the formula using the regression equation function on a graphing calculator.

**Activity 7: Round-Robin Tournaments (GLEs: 20, 22, 24, 25, 26, 27)**

The purpose of this activity is to use modeling or counting principles to determine answers to real-life problems. Examples:
- How many games need to be scheduled for six teams to play each other once?
- How many handshakes would take place among ten people if each person shakes hands with every other person?
- How many phone calls can be made between two people from among a group of five friends?
Encourage students to use various strategies for solving these problems. One technique is to model the situation by drawing and counting the diagonals in a polygon with the same number of sides as the number of teams or people. A second technique is to make lists showing all the possible combinations. Another technique is generating a formula by making a chart based upon how many handshakes would be needed for two people, three people, four people . . . n people. Prompt students to determine that reasoning is a valid process—each team plays every team except itself, but you need to divide by 2 to eliminate the duplicates (A playing B is the same as B playing A).

When determining the formula for answering each question, have students determine if the data is linear or not without using the graphing calculator. The regression equation function on the graphing calculator should be used to determine the formula only if the data is non-linear.

**Activity 8: Permutations and Combinations (GLE: 24)**

The purpose of this activity is for students to apply the concepts of permutation and combination covered in previous grades to geometric situations.

For example:

A. How many ways can 3 books be arranged on a shelf if they are chosen from a selection of 8 different books? *Solution: 336*

B. How many committees of 5 students can be selected from a class of 25?
*Sentence missing: Solution: 53, 130*

First, review simpler problems whose answers can be determined by making lists or tree diagrams. For example, how many different ways can you write the name of a triangle whose vertices are A, B, and C. The possibilities are

- ABC
- ACB
- BAC
- BCA
- CAB
- CBA

One way to think about this is that for any vertex that you start with, there are two different possible names. So three vertices times two names each is six possibilities. Another way to think about this is that there are 3 positions to fill when naming the triangle. You have 3 vertices from which to choose for the first position, but only 2 remain as choices for the second position. Once the second position is filled there is only one vertex remaining with which to fill the last position. Review with students that 3! or $3 \times 2 \times 1 = 6$ which is the same as the number of possible names. This is a concept taught in Algebra I. Give a few more examples in which the total number of combinations can be determined.
Relate the idea of determining how many choices one has to fill a position on the bookshelf to Problem A: How many ways can 3 books be arranged on a shelf if they are chosen from a selection of 8 different books?

There would be 8 ways to fill the first position, 7 ways to fill the second position, and 6 ways to fill the third position. $8 \times 7 \times 6 = 336$. In situations where order is important (e.g., ABC is different than ACB), the number of possibilities is called a permutation.

The permutation formula $P(n,r) = \frac{n!}{(n-r)!}$ was taught as part of the Algebra I curriculum. Review this formula with students and determine that the formula gives the same result as thinking about the number of positions and the choices that are available for filling each position.

For situations in which order is NOT important (i.e., ABC and ACB would be considered duplicates since they are the same three letters), the number of possibilities is called a combination. To know the number of combinations of 3 books that can be put on the shelf, take into account how many arrangements would be considered to be the same for each set of 3 books. This is $3!$ or 6, so dividing 336 by 6 is 56. There would be 56 different combinations to put on the shelf. In other words, one could display a different combination of 3 books for 56 days before he/she would have to repeat a set.

The combination formula $C(n,r) = \frac{n!}{(n-r)!r!}$ was also taught as part of the Algebra I curriculum. Review this formula with students to show that the formula results in the same answer (56) as the thinking and counting process.

Have students discuss problem B. First, have them determine if the problem requires a permutation or combination and then solve the problem accordingly. Ask students if a committee of John, Sue, and Mary is the same committee as Sue, Mary and John. (yes)

Provide students a variety of problems to work. It is better for some students to think through the position process. For those who have had more experience, the use of the formula is acceptable when solving such problems. Whether the formula is introduced and/or used should depend upon prior experience and knowledge of students in the class.

Introduce the class to circular permutations to answer such questions as, How many ways can $n$ people sit at a round table?

If one of the chairs is designated as the "head" of the table, then the answer is $n!$. Any of $n$ people sits at the head of the table, and the permutation proceeds in a clockwise direction. In this situation, it doesn't matter who sits at the head. In this case everyone could be sitting in the same relative order (ABCD, BCDA, CDAB, DABC for four people at the table) but seated in different chairs (A sits in position 1, then position 4, then position 3, then position 2 but A is always next to B, who is next to C, who is next to
D). Therefore, for \( n \) people there would be \( n \) duplicate arrangements. So \( n! \) divided by \( n \) duplicate arrangements results in \((n-1)!\) permutations if there is no designated head position in the circle.

Have students draw all arrangements for some simple problems to help them understand the process. For example, how many permutations are there for 3 people sitting at a round table? for 4 sitting at a round table? for 5? Then repeat the same process with the idea that one place is designated as the head position.

The following example shows a real-life application of a circle permutation.

*A disk jockey is setting up some CDs to play during his shift. He can put 6 different CDs on the tray. How many different ways can the discs be arranged?*

In this instance, once the discs are arranged in a circle, that same arrangement can be rotated. The discs are in different positions, but the arrangement is the same (If you label them ABCDEF and rotate it so that it is now FABCDE, the discs are still in the same relative order). Lead the students in a discussion to find that, in this case, 6 of the arrangements are the same so the permutation is \( \frac{6!}{6} \) or \((6-1)!\) possible arrangements of the discs.

Students should then generalize the concept so that any circular permutation without a fixed point is \((n-1)!\); with a fixed point, the permutation is \(n!\).

Examples

A. How many ways can 8 campers be seated around a campfire? Solution: 5040
B. How many ways can 3 books be placed on a shelf if chosen from a selection of 7 different books? Solution: 210
C. Find the total number of diagonals that can be drawn in an octagon. Solution: 20 (this is a combination taking 8 points 2 at a time—however, since 8 segments are the sides of the figure, those 8 must be subtracted from 28 which is the number obtained from the formula).
D. Given 7 distinct points in a plane, how many line segments will be drawn if every pair of points is connected? Solution: 21
E. Suppose there are 8 points in a plane such that no three points are collinear. How many distinct triangles can be formed with 3 of these points as vertices? Solution: 56
F. How many pentagons can be formed by joining any 5 of 11 points located on a circle? Solution: 462

Sample Assessments
General Assessments

- The student will create a variety of scenarios in which he/she make conjectures using inductive reasoning.
- The student will create portfolios containing samples of his/her activities. He/she should create some of his/her own patterns and exchange them with the other students in class and include them in his/her portfolio explaining whether the other students were able to determine their patterns and if they were able to figure out patterns made by others.
- The student will respond to journal prompts and explain his/her ideas. For instance:
  - Given the following pattern, explain how you would determine the formula for the pattern and how you would find the 35th term.
  - How are triangular, square, and rectangular numbers related to each other?
  - What is the difference between a permutation and a combination?

Activity-Specific Assessments

- **Activities 2 and 4:** Have the student create a variety of number or pictorial sequences. Each sequence should require the use of inductive reasoning to find the next number or picture in the sequence. Additionally, the student will indicate, orally or in writing, the process for generating the next item. The students will also state the rule for generating the nth term in each sequence.

- **Activity 3:** The student will use a graphing calculator to plot table entries for a given non-linear sequence in order to determine the regression equation for the data set.

- **Activity 7:** The student will participate in a simulation exercise to determine a tournament schedule for their district, regional, or state high school baseball team, basketball team, etc.
Geometry
Unit 2: Reasoning and Proof

Time Frame: Approximately two weeks

Unit Description

This unit introduces the development of arguments for geometric situations. Conjectures and convincing arguments are first based on experimental data, then are developed from inductive reasoning, and, finally, are presented using deductive proofs in two-column, flow patterns, paragraph, and indirect formats.

Student Understandings

Students understand the basic role proof plays in mathematics. Students come to distinguish proofs from convincing arguments. They understand that proof may be generated by first providing numerical arguments such as measurements and then replace the measurements with variables.

Guiding Questions

1. Can students develop inductive arguments for conjectures and offer reasons supporting their validity?
2. Can students develop short algorithmic-based proofs that generalize numerical arguments?
3. Can students develop more general arguments based on definitions and basic axioms and postulates?

Unit 2 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>Determine angle measurements using the properties of parallel, perpendicular, and intersecting lines in a plane (G-2-H)</td>
</tr>
<tr>
<td>17.</td>
<td>Compare and contrast inductive and deductive reasoning approaches to justify conjectures and solve problems (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td>19.</td>
<td>Develop formal and informal proofs (e.g., Pythagorean theorem, flow charts, paragraphs) (G-6-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>23.</td>
<td>Draw and justify conclusions based on the use of logic (e.g., conditional statements, converse, inverse, contrapositive) (D-8-H) (G-6-H) (N-7-H)</td>
</tr>
</tbody>
</table>
Sample Activities

Activity 1: Deductive Reasoning Skills (GLE: 17)

The teacher will remind students of the deductive reasoning skills used by Sherlock Holmes to solve mysteries by reading some excerpts (or provide excerpts for students to read) from Sherlock Holmes stories.

Ask students to work in small groups to solve a deductive reasoning or logic puzzle. (Teachers can search the Internet for logic puzzles or purchase puzzle magazines from local stores). Give groups no help in solving the puzzle. After the students have worked for a while, have class members discuss the strategies employed to solve the puzzle. Show them a solution using a chart to help solve the puzzle. Discuss the use of charts and how they might help. An example is provided below.

Frank’s Puzzle

Don, Frank, Jenny, and Ken each come from one state, either Alaska, Maine, Montana, or Oklahoma. Each speaks one primary language, either English, French, Russian, or Spanish. Each has one of four pets—a chinchilla, a dog, a hamster, or a turtle.

- Frank needed a language book to write to the Alaskan.
- The kid from Oklahoma has a mammal for her pet.
- The Alaskan found his pet outside his door in a snow bank.
- The French-speaking boy lives east of Oklahoma.
- The Russian-speaking boy wants to write to the kid from Montana, but he doesn’t speak his language.
- Don bought his pet in Peru.
- Ken does not own a hamster.
- The dog’s owner wrote a letter in Russian to the kid in Oklahoma, but she couldn’t understand it.
- Don had to travel west to meet Jenny.
- Frank is learning Spanish at school.

Solution: Don lives in Maine, owns a chinchilla, and speaks French; Frank lives in Montana, owns a turtle, and speaks English; Jenny lives in Oklahoma, owns a hamster, and speaks Spanish; Ken lives in Alaska, owns a dog, and speaks Russian.

Strategies: Discuss what skills students think are needed to solve the problem and what tools will help them solve the problem.

Once the better strategies have been determined, give the students another puzzle in their groups and allow them to work it.
Over a time period of one or two weeks, give the students puzzles of varying degrees of difficulty. Allow them to use help charts that you provide, but have them develop the ability to produce their own charts to facilitate their problem solving. Logic puzzles can be used throughout the year as filler and/or warm-up materials because this is a skill not easily learned by some students.

**Activity 2: Comparing Reasoning: Inductive vs. Deductive (GLE: 17)**

Have students evaluate reasoning skills and formulate their own examples. Review the definition of inductive logic.

**Judy’s Problem/Solution**

“My dad is in the Navy and he says that food is great on submarines,” offered Judy. “My mom,” added Bobbie, “works for the airlines and she says that airline food is notoriously bad.” “My mom is an astronaut trainee,” added Greer, “and she says that their food is the worst imaginable.” “You know,” concluded Judy, “I bet no life exists beyond earth!” Bobbie and Greer both looked at her, puzzled. “What?” “Sure,” explained Judy, “At extreme altitudes, food must taste so bad that no creature could stand to eat; therefore, no life exists out there.”

What do you think of Judy’s inductive reasoning? What possible other conjectures could be made? What are the problems with her conclusion?

After this discussion, define deductive reasoning and offer the following example for the students to analyze.

**Alex’s Grades**

Alex’s math teacher always tells him that homework is practice at home. She also tells him that the more he practices his math, the better his grades will be. Alex did all of his homework this week. When he gets to class before the test he tells his teacher, “I’m going to do well on the test today.”

What do you think of Alex’s deductive reasoning? Are there any problems with his conclusion?

The teacher should then lead a discussion about the differences between inductive and deductive reasoning. Students should be given other examples of reasoning and be asked to determine if the reasoning used was inductive or deductive.
Activity 3: Distinguishing Between Inductive and Deductive Reasoning (GLE: 17)

Have students visit http://www.sparknotes.com/math/geometry3/inductiveanddeductivereasoning/ for a presentation on inductive versus deductive reasoning. The site provides real-life examples of these types of reasoning and asks students to answer questions based on their reading. There are problems for students to solve as well. If teachers do not have the ability to take students to the lab, they can present the information on their class computers or print the information and give it to the students as worksheets.

Activity 4: Conditional Statements (GLE: 23)

Display the following conditional statement for all students to see: “If two angles have the same measure, then they are congruent.” Next, display the converse: “If two angles are congruent, then they have the same measure.” Lead a discussion about how these two statements are related.

Now, display the inverse of the conditional statement: “If two angles do not have the same measure, then they are not congruent.” Lead another discussion about how the conditional and the inverse statements are related.

Finally, display the contrapositive of the conditional: “If two angles are not congruent, then they do not have the same measure.” Lead a discussion about how the contrapositive is related to the converse.

Lead a discussion of the truthfulness of each statement.

After students have demonstrated an understanding of the relationships above, then display other conditional statements and have students work in pairs to write the converse, inverse, and contrapositive statements. Be sure to select conditional statements for which the converse is not a true statement.

Ask students to find examples of conditional statements in magazine or newspaper articles and discuss whether they are true or false. Have each student write and present the converse of his or her conditional statements to the class and explain why the converse is true or false.

Activity 5: Laws of Syllogism and Detachment (GLE: 23)

Display statements similar to the following: “All dogs are mammals. Buster is a dog.” These statements illustrate the law of detachment. Ask students to determine a logical conclusion from these statements. Have students rewrite the statements in a conditional format if it helps them to “see” the conclusion better. To illustrate the law of syllogism, present students with statements like the following: “If I study for tests, I will make good
grades. If I make good grades, I will be on the honor roll.” Ask students to form a logical conclusion based on these statements. Discussion should ensue about the validity (truthfulness) of these statements.

Present students with situations that do not lead to logical conclusions. Ask students to write their own pairs of conditionals that lead to logical conclusions and pairs that do not lead to logical conclusions. This activity will help students develop their deductive reasoning skills.

Have students apply the laws of syllogism and detachment to algebraic and geometric concepts in preparation for proofs.

Activity 6: Algebraic Proofs (GLE: 19)

In this activity, have students work in cooperative groups to put together proofs for basic algebra concepts. Give groups completed proofs with all of the statements and reasons, but do not have the steps in logical order. One way to present this task to the students is to have the statements and reasons on strips of paper that the students can physically arrange in correct order and then copy the final result on paper.

Once students have adjusted to organizing the proofs, introduce proofs with unnecessary information. Require that students use only information that is relevant to the proof and organize the information into a logical order. Provide students with the opportunity to progress from basic algebraic proofs to basic geometric proofs based on algebraic concepts (definition of congruence, angle and segment addition postulates, properties of equality).

Activity 7: Proofs (GLE: 19)

Proofs should emphasize flow proof or paragraph proof rather than two-column proof. More emphasis should be placed on providing a convincing, easy-to-follow argument with reasons than on using one particular format. The content for these proofs should focus on basic geometric concepts (segment and angle addition, congruent segments and angles). While this content was also covered in Activity 6, these proofs are different because students have to come up with the statements and reasons. Students must determine the arguments and reasons with their classmates. Facilitate students’ work with proofs by doing the following:

• Have students work in small groups of three to four students.
• Working together in the group, have students complete a proof in three different ways, then write their proofs on one page and turn them in.
• Look for correct proofs. Some groups will work together, while others will write individual proofs and then compare. Encourage them to discuss ideas with each other.
• Choose three different groups to write a particular (correct) proof on the board. As a class, discuss variations and similarities of the three proofs, and talk about extra steps that could be added or omitted.

Activity 8: Fun with Angles (GLEs: 11, 19, 23)

Review the relationships among angles formed by the intersection of two parallel lines and a transversal that were learned in grade 8. Provide students with a graphic similar to Diagram 1 in which lines \(a\) and \(b\) are parallel. First, provide a number that represents the measure of angle 1. Have students find the measures of all the other numbered angles in the diagram and provide a justification for each measurement found (e.g., if the measure of angle 1 is 105°, the measure of angle 5 is 105° because angles 1 and 5 are corresponding angles).

Next, have students provide a convincing argument that pairs of angles are either congruent or supplementary (e.g., given that lines \(a\) and \(b\) are parallel, prove that angles 1 and 7 are supplementary), but without using angle measures. (Solution: If lines \(a\) and \(b\) are parallel, then angles 1 and 5 are congruent corresponding angles. Angles 5 and 7 are supplementary because they form a linear pair. If angles 5 and 7 are supplementary and angle 1 is congruent to angle 5, then angles 1 and 7 must also be supplementary since angles which are congruent can be substituted for one another.)

Slightly more difficult proofs can be devised using diagrams similar to Diagram 2.

Use activities that require students to provide proofs or convincing arguments for answers throughout the year.
Sample Assessments

General Assessments

- The student will answer journal prompts that include:
  - Comparing inductive and deductive reasoning.
  - Describing a situation in which he/she had several experiences that led you to make a true conjecture. Then describe a situation where he/she had several experiences that led to a false conjecture.
  - Responding to an advertisement, such as the following: “Those who choose Tint-and-Trim Hair Salon have impeccable taste; and you have impeccable taste” shows misuse the Law of Detachment to make the reader come to an invalid conclusion.
    a. What conclusion does the ad want imply?
    b. Write another example that illustrates incorrect logic.

- The student will create a portfolio containing samples of his/her activities. For instance, the student will select the logic puzzle he/she liked best, explain how it was solved, and why he/she likes it.

- The student will write the inverses, converses, and contrapositives of given conditional statements, organize information for a proof, write his/her own proofs for basic algebraic and geometric concepts, and draw conclusions based on the laws of syllogism and detachment.

Activity-Specific Assessments

- Activity 1: The student will create his/her own logic problems. The students should then solve each others problems and the student will turn in the solutions for assessment.

- Activity 2: The student will find instances in real-life where logical conclusions have been made. He/she can use newspapers, magazines, experiences at home, etc. and will write a paragraph explaining whether the logic used was inductive or deductive and if the conclusions are true or false. In this explanation, the student will demonstrate a concrete understanding of the difference between inductive and deductive reasoning.

- Activity 2: The student will read a book or watch a movie or TV show in which deductive logic is used to solve a mystery. He/she will detail the facts used in the deductive process in a short synopsis of the book or movie.

- Activity 5: The student will write proofs on his/her own using basic algebra concepts. The student will explain the reasons for each step in the process of solving the problem.
• **Activities 5 and 6**: The teacher will provide students with proofs—some that are accurate and some that have flaws. The student will evaluate the proofs and identify and correct any flaws that exist.

• **Activity 7**: The teacher will provide students with one measurement in a diagram using parallel lines and transversals (possibly three parallel lines and one or two transversals). The student should find all the missing angle values in the diagram and provide an explanation of how each value was determined.
Geometry
Unit 3: Parallel and Perpendicular Relationships

Time Frame: Approximately three weeks

Unit Description

This unit demonstrates the basic role played by Euclid’s fifth postulate in geometry. The focus is on the basic angle measurement relationships for parallel and perpendicular lines, the equations of lines that are parallel and perpendicular in the coordinate plane, and proving that two or more lines are parallel using various methods including distance between two lines.

Student Understandings

Students should know the basic angle measurement relationships and slope relationships between parallel and perpendicular lines in the plane. Students can write and identify equations of lines that represent parallel and perpendicular lines. They can recognize the conditions that must exist for two or more lines to be parallel. Three-dimensional figures can be connected to their 2-dimensional counterparts when possible.

Guiding Questions

1. Can students relate parallelism to Euclid’s fifth postulate and its ramifications for Euclidean Geometry?
2. Can students use parallelism to find and develop the basic angle measurements related to triangles and to transversals intersecting parallel lines?
3. Can students link perpendicularity to angle measurements and to its relationship with parallelism in the plane and 3-dimensional space?
4. Can students solve problems given the equations of lines that are perpendicular or parallel to a given line in the coordinate plane and discuss the slope relationships governing these situations?
5. Can students solve problems that deal with distance on the number line or in the coordinate plane?
Sample Activities

Activity 1: Ladders and Saws (GLEs: 10, 11)

Instructions for the Ladder and Saws activity can be found in many texts and on the Web. For example, see http://library.thinkquest.org/28318/ladders.html. In this activity, students use parallel segments in hands-on activities to discover various relationships between angles, segments, and triangles.

Below is a list of some of the possible concepts/principles that students may “discover” by engaging in the Ladders and Saws activity (as listed on the website http://library.thinkquest.org/28318/ladders.html).

- The sum of the angle measures of any triangle is equal to 180°.
- Vertical angles are congruent.
- Linear pairs are supplementary.
- Alternate interior angles are congruent when lines are parallel.
- Corresponding angles are congruent when lines are parallel.
- Same-side interior angles are supplementary when lines are parallel.
- The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles of the triangle.
• Two lines parallel to a third line are parallel to each other.
• Opposite sides of a parallelogram are parallel and congruent.
• Opposite angles of a parallelogram are congruent.
• Adjacent angles of a parallelogram are supplementary.
• The segment joining the midpoints of two sides of a triangle is parallel to the third side and has a length equal to half the length of the third side.
• The ratio of the perimeters of two similar triangles is the same as the scale factor of the similar triangles.
• The sum of the exterior angle measures of any convex polygon is 360°.
• The sum of the angle measures of a quadrilateral is 360°.
• The sum of the angle measures of a hexagon is 720°.

Teachers should insure that students understand the relationships among the angles formed by the parallel lines and transversals first. Have students discuss the other concepts mentioned above and retain this listing of relationships so that they may refer to it during discussions in subsequent units that refer to these concepts.

Activity 2: Slopes of Perpendicular Lines (GLEs: 10, 11, 22)

Have students work in pairs. Provide each pair of students with three graphs of lines, each on a separate coordinate grid system. On one set of axes, have a graph of parallel lines; on the other two sets of axes have single lines with different slopes and y-intercepts. Be sure to use a different set of lines for each pair of students. Provide at least two points on each line. Review how to find the slope of a line from two points, and then have students determine the slope of each of the lines provided.

Next, have students carefully fold each of the single lines onto itself and crease the paper along the fold line. Have students measure the angle formed by the line and the “crease” line to confirm that it is a right angle. Be sure to engage students in a discussion that helps them see that the “crease” line is perpendicular to the original line. Students should develop a convincing argument that the “crease” line is perpendicular to the original lines (i.e., The two angles are congruent because they are the same size since the angles “match” when folded over one another. Because a line measures 180 degrees, the measures of the two angles are 90 degrees each).

Students will then determine the slope of the “crease” line and compare it to that of the original line. All data from the class should be recorded in a chart. The chart should include a column for the slope of the original line and a column for the slope of the “crease” line. Using the class data, student pairs should make a conjecture about the slope of perpendicular lines.

Using the graph with the given parallel lines, have students discuss how the slopes of the parallel lines are related and whether or not the “crease” is perpendicular to one or both of the given lines. This should lead to a discussion about the theorem that states if a line...
is perpendicular to one of two parallel lines, then it is perpendicular to the other.

**Activity 3: Parallel and Perpendicular Lines (GLE: 6)**

Provide students with several equations of pairs of lines that are parallel and several equations of pairs that are perpendicular, but don’t give them the relationships. Have students graph the lines and determine the characteristics of the equations of the lines that are parallel and the characteristics of those that are perpendicular (i.e., parallel lines have the same slopes and different y-intercepts, perpendicular lines have slopes that are opposite reciprocals of one another). As an alternative, have students use a computer software program like *Geometer’s Sketchpad* to draw a pair of perpendicular lines and a pair of parallel lines. Have the program generate the equations of those lines and then determine the characteristics of these equations.

Once the characteristics are determined, review with students the process for developing the equation of the line if two points on the line are given. Provide students with graphs of lines that are parallel or perpendicular (several of each). Have students apply the characteristics of parallel or perpendicular lines to write the equations for the given lines. Next, have students write the equation of lines that are parallel or perpendicular to a line through a given point on the line.

Examples:

Given that a line passes through \((-2,3)\) and \((4,6)\), write the equation of a line that is parallel to the given line and passes through \((1,–2)\). Write the equation of the line that is perpendicular to the original line through \((1,–2)\). *Solution: Parallel:*

\[
y = \frac{1}{2}x - \frac{5}{2}; \text{ Perpendicular: } y = -2x.
\]

To end the activity, have students write equations of any two lines that are parallel and any two lines that are perpendicular without any given information.

**Activity 4: Proving Lines are Parallel (GLEs: 10, 11, 19)**

In this activity, give students diagrams of lines that are parallel and diagrams of lines that are not parallel. Lead a discussion to determine what characteristics of parallel lines will guarantee that two lines are parallel.

Have students form conjectures that lead to the converses of the parallel lines theorems (e.g., if alternate interior angles are congruent when a transversal intersects two lines, then the two lines are parallel). Ask students to provide a convincing argument for the postulate concerning corresponding angles since this cannot be proven. Provide opportunities for students to prove the other theorems which are based on the postulate for corresponding angles. Allow students to initially use angle measures to write proofs for specific sets of lines to prove these theorems, but also require them to use general
proofs that prove lines parallel through generalities. These proofs can take any form (informal, paragraph, two-column, flow).

Provide diagrams of two lines that are perpendicular to one line. Have students form a conjecture that if two lines are perpendicular to the same line, they must be parallel. Then, have students write a proof of this theorem.

**Activity 5: Distance in the Plane (GLEs: 1, 12, 16)**

Have students explore the distance between two points in the rectangular coordinate system. Give attention to the idea that distance in the plane between two points can be thought of as the length of a hypotenuse of a right triangle. Thus, the Pythagorean theorem can be used to determine these distances. Have students apply the concept of distance on a number line to find the length of the legs of the right triangle. Once students understand how the Pythagorean theorem applies to distance on a coordinate plane, guide students as they develop the formula for distance on the coordinate plane through the use of arbitrary points \((x_1, y_1)\) and \((x_2, y_2)\). When using the Pythagorean theorem and the distance formula, have students simplify radical solutions as well as use the calculator to estimate the solution.

**Activity 6: Parallel Lines and Distance (GLEs: 10, 16)**

Provide different sets of two lines. Some sets should be parallel; others should not be parallel. These lines should be drawn on lineless paper. Ask students how they visually determine which of the sets of two lines are parallel. Have a discussion which leads to an understanding that parallel lines are always the same distance apart.

Lead students in a discussion about the definition of distance between a point and a line or distance between two parallel lines. One way to do this is to draw two parallel lines on the board and use a ruler to determine distances. Use a drawing program, such as Geometer’s Sketchpad®, as an alternate way to demonstrate the same concept. The discussion should reveal that the distance is always the shortest line segment between two points (or the shortest distance between a point and a line). Have students realize that distance between a point and a line is the same as the length of a line segment which starts at the point and is perpendicular to the line. To find the distance between two parallel lines, identify a point on one of the lines and draw a segment from this point perpendicular to the second line.

Give students diagrams on the coordinate plane and ask them to find the distance between lines and points not on the lines. Review the concept of the Parallel Postulate here (If there is a line and a point not on the line, then there is exactly one line that can be drawn through the given point that is parallel to the given line).
Have students apply the concept of distance between a point and a line to polygons by relating the distance to the altitude of a triangle, the measure of the side of a rectangle or square, and the height of various geometric figures in both 2-D and 3-D space. This establishes correct understandings necessary for the concepts necessary for finding area, surface area, and volume.

**Activity 7: Parallel Line Facts (GLE: 10)**

Have students draw a line through one vertex of a triangle so that the line is parallel to a side of the triangle. Have students write a proof (using parallel line relationships from Unit 2) to show that the sum of the angles in a triangle is $180^\circ$.

Use the same diagram to show that the measure of an exterior angle in a triangle has the same measure as the sum of the two remote interior angles.

Have students investigate the area of different triangles formed between two parallel lines by moving one vertex along one of the parallel lines. Students should recognize that the height of each triangle is always the same since the distance between the parallel lines will not change. Since the base length doesn’t change, students should realize that the areas are the same.

<table>
<thead>
<tr>
<th>Sample Assessments</th>
</tr>
</thead>
</table>

**General Assessments:**

- The student will create a portfolio containing samples of work completed during activities. For instance, he/she could include the graphs from activity 2 and explain what happened in the activity and what was learned from the activity.
- The student will respond to journal prompts that include:
  - Describing at least three different ways to prove two lines are parallel.
  - Explaining how to write the equation of a line perpendicular to $y = -\frac{2}{3}x + 5$ through the given point (-4,6).
  - Explaining the relationship between the Pythagorean theorem and the distance formula for distance on the coordinate plane.
- The student will create a “scrapbook” of pictures taken in a real-world setting (i.e. railroad tracks) that depict parallel and perpendicular lines. This scrapbook will
include pictures and indicate how the items in the picture demonstrates the term chosen. The student will have a minimum of three pictures for each term. See the rubric in the Resources section for more information.

Activity-Specific Assessments

- **Activity 1**: The teacher will provide the student with nets or diagrams formed by intersecting lines (parallel and nonparallel) and a minimal number of angle measures for the diagram. The student will calculate the missing angle measures using either the formula $S = 180(n - 2)$ or the angles created by transversals that intersect parallel lines.

- **Activity 2**: The teacher will provide the student with several sets of graphs of parallel and almost parallel lines that are drawn on coordinate graph paper. The student will use slope to determine if the lines are parallel.

- **Activity 3**: The teacher will give the student a line and a point not on the line drawn on a sheet of graph paper with no axes. The student will draw and label the $x$ and $y$-axis anywhere on the coordinate plane that he/she chooses. Based on where the $x$ and $y$-axes are drawn, the student will then
  - find the slope of the given line and write the equation of the given line.
  - write the equation for a line which passes through the given point and is parallel to the given line.
  - write the equation of the line which is perpendicular to the given line and passes through the given point.

A “What’s My Line?” guide for this assessment is provided at the end of this unit.
## Resources

### Parallel and Perpendicular Lines Scrapbook

<table>
<thead>
<tr>
<th>CATEGORY</th>
<th>4 points</th>
<th>3 points</th>
<th>2 points</th>
<th>1 point</th>
<th>0 points</th>
<th>Score</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quantity (24)</strong></td>
<td>Minimum of 3 photos per term (Parallel or Perpendicular)</td>
<td>Only two photos/pictures per term</td>
<td>Only one photo/picture per term</td>
<td>Only one picture to demonstrate both terms</td>
<td>No photos or pictures</td>
<td>x 6</td>
<td></td>
</tr>
<tr>
<td><strong>Quality (24)</strong></td>
<td>Photos are of an excellent quality; clear; description is written clearly</td>
<td>Photos/pictures are of good quality; description is clear but missing some elements</td>
<td>Photos/pictures are grainy; term is not clearly depicted in picture; description is vague</td>
<td>Photos/pictures do not depict term at all; description only gives definition</td>
<td>No description given</td>
<td>x 6</td>
<td></td>
</tr>
<tr>
<td><strong>Title Page (8)</strong></td>
<td>Excellent quality; typed; includes project title, date, and class period</td>
<td>Typed; missing date or class period</td>
<td>Handwritten with all information; or typed and missing date and class period</td>
<td>Handwritten and missing date and class period; missing title (typed with all other info)</td>
<td>No title page</td>
<td>x 2</td>
<td></td>
</tr>
<tr>
<td><strong>Reflection (12)</strong></td>
<td>Typed; tells what the student learned from project; grammatically correct</td>
<td>Typed; 1-2 grammar errors; some evidence of learning</td>
<td>Handwritten; 3-4 grammar errors; vague evidence of learning</td>
<td>Handwritten; 5-6 grammar errors; little to no evidence of learning</td>
<td>No reflection</td>
<td>x 3</td>
<td></td>
</tr>
<tr>
<td><strong>Neatness/Creativity (8)</strong></td>
<td>Typed; clean; neatly bound pages; original project title; attractive; etc.</td>
<td>Typed; project name not original; some pages loose</td>
<td>Handwritten; project is less than attractive;</td>
<td>Dirty, crumpled pages; if handwritten there are scratchouts or places with liquid paper</td>
<td>Pages are not bound</td>
<td>x 2</td>
<td></td>
</tr>
<tr>
<td><strong>Timeliness (8)</strong></td>
<td>Turned in on time</td>
<td>Turned in one day late</td>
<td>Turned in two days late</td>
<td>Turned in three days late</td>
<td>Turned in more than three days late</td>
<td>x 2</td>
<td></td>
</tr>
</tbody>
</table>
What’s My Line?

Directions: Using the line and point given to you on the graph paper, complete each of the following. Be sure to show all necessary work and label all necessary lines and points.

1.) Draw and label the x and y-axes where you want them to go.

2.) Identify and label two points on your line. Using the points you identified, find the slope of your given line. Write the slope on the line as $m=_____$. 

3.) Write the equation of the given line in slope-intercept form. Write the equation on the line.

4.) Write the equation of the line that is parallel to the given line and passes through the given point that is not on the given line. Graph this line and write the equation on the line.

5.) Write the equation of the line that is perpendicular to the given line and passes through the given point that is not on the given line. Graph this line and write the equation on the line.
Geometry
Unit 4: Triangles and Quadrilaterals

Time Frame: Approximately five weeks

Unit Description

This unit introduces the various postulates and theorems that outline the study of congruence and similarity. The focus is on similarity and congruence treated as similarity with a ratio of 1 to 1. It also includes the definitions of special segments in triangles, classic theorems that develop the total concept of a triangle, and relationships between triangles and quadrilaterals that underpin measurement relationships.

Student Understandings

Students should know defining properties and basic relationships for all forms of triangles and quadrilaterals. They should also be able to discuss and apply the congruence postulates and theorems and compare and contrast them with their similarity counterparts. Students should be able to apply basic classical theorems, such as the isosceles triangle theorem, triangle inequality theorem, and others.

Guiding Questions

1. Can students illustrate the basic properties and relationships tied to congruence and similarity?
2. Can students develop and prove conjectures related to congruence and similarity?
3. Can students draw and use figures to justify arguments and conjectures about congruence and similarity?
4. Can students state and apply classic theorems about triangles, based on congruence and similarity patterns?

Unit 4 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Simplify and determine the value of radical expressions (N-2-H)(N-7-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>Write the equation of a line parallel or perpendicular to a given line through a specific point (A-3-H) (G-3-H)</td>
</tr>
</tbody>
</table>
### Sample Activities

**Teacher Note: Before beginning this unit, the teacher should make sure the students have a good understanding of the Angle Sum Theorem and Exterior Angle Theorem. These theorems should have been discussed during Unit 2 while working with properties of parallel lines. Students should also be able to classify triangles according to their side and angle measures. If necessary, the teacher may take a day or two to review these concepts; however, this should be kept to a minimum.**

**Activity 1: Analyzing Isosceles Triangles (GLE: 1, 10, 16,)**

**Teacher Note: Safety compasses can be used in schools where sharp, pointed instruments are prohibited. Techniques for using patty paper to duplicate segments, construct perpendicular lines, angle bisectors, etc. can be found in Patty Paper Geometry by Michael Serra (Key Curriculum Press).**

Have students use patty paper or tracing paper to draw an acute, isosceles triangle. Have them start by drawing an acute angle and label it \( C \). Then they should mark equal lengths on each side of the angle, label them \( A \) and \( B \), and draw \( AB \). Have students use patty paper constructions, compass/straight edge constructions, or measurements with a ruler to mark equal lengths. Have students fold the triangle in half so that the equal sides lie on top of each other. Have students make observations about base angles \( A \) and \( B \). Repeat the activity with obtuse and right isosceles triangles as well. Have a class discussion in which students form conjectures that lead to the Isosceles Triangle Theorem (If two sides...
of a triangle are congruent, then the angles opposite those sides are congruent.) and its converse.

Have students practice their algebra skills to find the measures of sides and angles of isosceles triangles when given information about the isosceles triangles.

For example:

A. In isosceles triangle \( \triangle ABC \) with base \( \overline{BC} \), \( m\angle ABC = 5x - 4^\circ \), and \( m\angle ACB = 7x - 20^\circ \). Find the measure of each angle.
   
   **Solution:** \( x = 8 \), \( m\angle ABC = 36^\circ \), \( m\angle ACB = 36^\circ \), and \( m\angle BAC = 108^\circ \).

B. In isosceles triangle \( \triangle DEF \), \( \angle F \) is the vertex angle. If \( DE = 5x \) inches, \( EF = 4x - 3 \) inches, and \( DF = 2x + 7 \) inches, find the length of the base.
   
   **Solution:** \( x = 5 \), and \( DE = 25 \) inches.

C. \( \triangle ABC \) has vertices \( A(2,5), B(5,2), \) and \( C(2,1) \). Use the distance formula to show that \( \triangle ABC \) is an isosceles triangle and name the pair of congruent angles.

   **Solution:** \( AB = 3\sqrt{2}, BC = 3\sqrt{2}, AC = 6, \overline{AB} \cong \overline{BC}, \) and \( \angle A \cong \angle C. \) All distances are in linear units.

**Activity 2: Congruent Triangles (Using Technology) (GLE: 10)**

Have students quiz themselves and track results concerning terminology used with congruent triangles at the website [http://www.quia.com/jq/13397.html](http://www.quia.com/jq/13397.html). This activity is an assessment of the students’ prior knowledge. Repeat the activity later in the unit to see if gains were made.

**Activity 3: Corresponding Parts (CPCTC) (GLEs: 18)**

*Note: While this activity does not ask students to find the measures of segments or angles, it does require them to determine corresponding parts of congruent triangles. This is a skill necessary when determining corresponding sides to write proportions for similarity.*

The focus of this lesson is to make students aware of correct ways to name congruent triangles to preserve corresponding parts. Have students work in small groups. Give each group several diagrams of pairs of congruent triangles. Have students measure sides and angles to determine congruent, corresponding parts. Lead a discussion about writing congruence statements and focus on writing letters in the correct order. Provide students with congruence statements such as \( \triangle ABC \cong \triangle XYZ \) and ask them to name the corresponding angles and sides. Additionally, instruct students to find equivalent
congruence statements for figures (e.g., for the example shown, \( \triangle BAC \cong \triangle YXZ \) is an equivalent statement).

**Activity 4: More about Congruent Triangles (Using Technology) (GLEs: 10, 19)**

Using Geometer’s Sketchpad® or similar geometry software, give students a set of three lengths for line segments that can be used to form a triangle. Have students construct the triangle. Next, have students compare their constructions. They should make a conjecture about the triangles created (e.g., the triangles are congruent).

Next, have students repeat the activity using two side lengths and an included angle. Again, have students conjecture the relationship between the constructed triangles. Repeat the activity using two angles and an included side and then again using two angles and a non-included side.

Ask students to use their constructions to help them develop convincing arguments for the postulates discovered in this activity (SSS, SAS, ASA, and AAS). If the class does not have access to geometry software, have students use straws cut to certain lengths and protractors. Provide alternative ways for students to draw the given triangles by hand if materials are not accessible (compass/straightedge or patty paper).

**Activity 5: Are They Congruent? (GLEs: 10)**

Provide students with the measures of two sides and a non-included angle for a triangle or the measures of three angles and no sides. Have students construct a triangle using the given measures (either with or without technology). Next, have students compare their constructions. Have students make a conjecture about the relationships of these constructed triangles. Repeat this activity with several sets of SSA or AAA measures. Students should make a conjecture about whether SSA and AAA can be used to justify two triangles being congruent.

**Activity 6: Proving Triangles Congruent (GLEs: 17, 19, 23)**

Have students work in pairs. Provide students with diagrams which include congruent triangles. Each group should have different diagrams. For instance, students may be given a diagram of a rectangle with both diagonals drawn. Ask students to prove two triangles of the triangles in the diagram congruent. Allow students to use various methods of proof: two-column, flow, or paragraph.

Have groups share their proofs with the class. Encourage class members to question each other if they believe key aspects have been omitted. To end the activity, have students employ techniques used in class to prove two triangles from a diagram congruent. This should be individual work to show that students have mastered the skill.
Example:
Given: $X$ is the midpoint of $BD$
$X$ is the midpoint of $AC$
Prove: $\triangle DXC \cong \triangle BXA$

Activity 7: Altitudes, Angle Bisectors, Medians, and Perpendicular Bisectors of a Triangle (GLE: 10)

Give each student four sheets of patty paper. Instruct students to draw four different triangles—one on each piece of patty paper. Make sure that at least one of the triangles is not a right triangle and also not an obtuse triangle. Have students label each sheet of patty paper with either perpendicular bisectors, medians, angle bisectors, or altitudes.

Provide students with the definition of angle bisector. Have students construct the angle bisectors for all the angles in one of their triangles. This may be done via patty paper folding, measuring, or with the use of a drawing program such as the Geometer’s Sketchpad®. The purpose of this activity is to have students learn the definition by having to create angles of equal measure. Ask students to share their work with other class members. If done properly, the angle bisectors will intersect in one point.

Provide students with the definition of median. Have students construct the three medians in one of their triangles. This may be done via patty paper folding, measuring, or with the use of a drawing program such as the Geometer’s Sketchpad®. The purpose of this activity is to have students learn the definition by having to first find the midpoint of a side and then draw a segment from the midpoint to the opposite vertex in the triangle. Ask students to share their work with other class members. If done properly, the medians will intersect in one point.

Provide students with the definition of perpendicular bisector of a segment. Have students construct the perpendicular bisectors for all sides in one of their triangles. This may be done via patty paper folding, measuring, or with the use of a drawing program such as the Geometer’s Sketchpad®. The purpose of this activity is to have students learn the definition by having to locate the midpoint of a side and then draw a line through the midpoint so that the line is perpendicular to the side of the triangle. Ask students to share their work with other class members. If done properly, the perpendicular bisectors of the three sides of the triangle will intersect in one point.

Provide students with the definition of altitude in a triangle. Have students construct the three altitudes in one of their triangles. (Note: Have the students use an acute scalene triangle on their first try as the altitudes for this type of triangle will lie in the interior of the triangle.) This may be done via patty paper folding, measuring, or with the use of a drawing program such as the Geometer’s Sketchpad®. The purpose of this activity is to have students learn the definition by creating a line that passes through a vertex of the triangle and is perpendicular to the opposite side. Ask students to share their work with
other class members. If done properly, the altitudes will intersect in one point. After students can successfully perform this activity with an acute, scalene triangle, show students how to create altitudes for obtuse and right triangles. Many students will have trouble visualizing where the altitudes are in obtuse triangles and will require assistance from the teacher and much practice.

*Teacher Note: Instructions for using patty paper to fold segments in this activity can be found in Patty Paper Geometry by Michael Serra (Key Curriculum Press). It is not recommended that these constructions be made with a compass/straightedge as students seldom remember the construction steps. If compass/straightedge constructions are used, time must be taken to explain and demonstrate how the constructions relate to congruent triangles (i.e., the construction of a perpendicular bisector of a segment is based on the creation of two triangles by SSS). Patty paper constructions or use of the Geometer’s Sketchpad are much more intuitive for students, and their use does not present safety issues.*

Have students practice constructing their own altitudes, perpendicular bisectors, medians, and angle bisectors to help internalize the definitions.

*Extensions:* An acute, scalene triangle works best for these activities. Have students draw the triangle on a sheet of patty paper. The triangle should be as large as possible.

Have students fold or draw all angle bisectors in a triangle. Tell them the name of the common intersection point for the three angle bisectors in a triangle is called the *incenter.* Have students measure the distance from the incenter to each side. This reinforces the concept of distance between a point and a line. *These distances should be the same, indicating that the incenter is the center of a circle which can be inscribed in the triangle.* Have students use a compass to draw the inscribed circle.

Have students fold or draw all the perpendicular bisectors in a triangle. Tell them the name of the common intersection point for the three perpendicular bisectors in a triangle is called the *circumcenter.* Have students measure the distance from the circumcenter to each vertex of the triangle. This reinforces the concept of distance between two points. These distances should be the same indicating that the circumcenter is the center of a circle which passes through each vertex of the triangle. Have students use a compass to draw the circumscribed circle.

Have students fold or draw the three medians in a triangle. Tell them the name of the common intersection point for the three medians is called the *centroid* and is the center of gravity for the triangle. Have students transfer the location of the three vertices of the triangle and the centroid to a sheet of cardstock. (An old manila file folder works well, too.) This can be done by placing the sheet of patty paper on the card stock and making an indentation with a pencil or pen point through the patty paper onto the card stock. Have the students use a straight edge to draw the sides of their triangle on the cardstock and then cut it out with scissors. If done properly, the triangle should balance when the centroid is placed on the lead end of a sharpened pencil. Use the eraser end, if needed.
Position the pencil at a location other than the centroid, and the triangle will tilt to one side and fall off. It will not stay balanced.

Some students may want to create art designs using triangles and the inscribed and/or circumscribed circles for a portfolio entry.

**Activity 8: Altitudes, Medians, and Perpendicular Bisectors on the Coordinate Plane (GLEs: 6, 9)**

Provide students with information that allows them to graph triangles on a coordinate plane. Have them draw the medians, perpendicular bisectors, and altitudes of those triangles. Ask students to write equations that represent those segments. Writing equations reinforces skills learned in Algebra.

**Examples:**

A. $\triangle ABC$ has vertices $A(-3,10)$, $B(9,20)$, and $C(-2,21)$. Find the coordinates of $P$ such that $\overline{CP}$ is a median of $\triangle ABC$. Determine if $\overline{CP}$ is an altitude of $\triangle ABC$.

**Solutions:** $P$ is at $(3,15)$; $\overline{CP}$ is an altitude of $\triangle ABC$.

B. The following equations intersect to form a triangle. Identify the vertices of the triangle.

\[
\begin{align*}
2x - y &= 6 \\
3x + 2y &= -2 \\
9x - 2y &= 26
\end{align*}
\]

Draw one of the perpendicular bisectors in the triangle and identify the slope and point used to draw it. Then write the equation for that perpendicular bisector.

**Solutions:** Vertices are $(-2,2)$, $(4,5)$, and $(2,-4)$. Students should have one of the following for the point, slope, and equation:

A. $(1,3.5)$, $m = -2$, $y = -2x + 5.5$

B. $(0,1)$, $m = \frac{5}{2}$, $y = \frac{5}{2}x - 1$

C. $(3,\frac{7}{2})$, $m = -\frac{2}{7}$, $y = -\frac{2}{7}x + \frac{7}{6}$

**Activity 9: More on Angle Bisectors, Medians, and Perpendicular Bisectors of a Triangle (GLE: 10)**

Have students complete this activity with a partner. They will need an automatic drawer.

1. Using an automatic drawer, such as that found in *Geometer’s Sketchpad*®, draw scalene triangle $ABC$ and measure the lengths of $\overline{AB}$ and $\overline{AC}$.
2. Construct $m$, the angle bisector of $\angle BAC$.
3. Construct the midpoint $D$ and the perpendicular bisector of $\overline{BC}$.
4. Draw the median from point $A$ to $\overline{BC}$.
5. Move point $A$ until the angle bisector, perpendicular bisector, and the median coincide. Record the lengths of $\overline{AB}$ and $\overline{AC}$.
6. Drag point $A$ to find two other positions for point $A$ in which angle bisector, perpendicular bisector, and the median coincide. Again, record the lengths of $\overline{AB}$ and $\overline{AC}$.

Ask students to make a conjecture about $\triangle ABC$ when the angle bisector of $\angle BAC$, the median from $A$ to $\overline{BC}$, and the perpendicular bisector of $\overline{BC}$ coincide. Have students write a proof to show that if a segment is a median and an angle bisector in the same triangle, then the triangle is isosceles.

**Activity 10: Proving Right Triangles Congruent (GLEs: 10, 19, 23)**

Give students diagrams showing pairs of right triangles with different segments and acute angles marked congruent. Help them to determine which pairs of the triangles are congruent and be able to explain why they are congruent. Then have the students write congruence proofs for these triangles using methods already discussed like SAS, AAS, and ASA. Connect these proofs and methods to the LL, HA, and LA theorems for right triangles. Also, introduce the HL postulate. Provide opportunities for students to write proofs for different sets of right triangles formed by perpendicular bisectors and altitudes in triangles as well.

**Example:**

Given: $\overline{BD}$ is a perpendicular bisector in $\triangle ABC$

Prove: $\triangle ABD \cong \triangle CBD$

**Solution:** Since $\overline{BD}$ is a perpendicular bisector, $\angle BDA$ and $\angle BDC$ are right angles because perpendicular lines form 4 right angles. That makes $\triangle ABD$ and $\triangle CBD$ right triangles. By the definition of perpendicular bisector, $D$ is the midpoint of $\overline{AC}$. By the definition of midpoint, $\overline{AD} \cong \overline{CD}$. Also, $\overline{BD} \cong \overline{BD}$ because congruence of segments is reflexive. Therefore, since two pairs of legs of $\triangle ABD$ and $\triangle CBD$ are congruent, $\triangle ABD \cong \triangle CBD$ by LL.

**Activity 11: Inequalities for Sides and Angles in a Triangle (GLE: 1, 10, 16)**

Have each student draw a triangle on a sheet of patty paper and label it $\triangle ABC$. Make sure each student has his/her name on the sheet. Ask students to exchange triangles so that they do not measure their own triangles. Instruct students to measure each side and each angle and record the measurements in a chart on a different sheet of paper. Have
students record the name of the student whose triangle they measured. Have students make observations about the angle opposite the longest side in relation to the other two angles, and the side opposite the smallest angle in relation to the other two sides. Ask students to repeat the activity with a different triangle. After each student has measured two different triangles, have students work in small groups to discuss their observations and form conjectures concerning the sides and angles in a triangle. Ask students to test their conjectures by trying to counterexamples.

After this measuring activity, have students solve problems that require an understanding of this concept. Provide students with diagrams showing triangles and their angle measures. Ask students to list the sides in order from longest to shortest or shortest to longest. Provide other diagrams showing triangles and the lengths of the sides. Have students list the angles in order from least to greatest or vice versa. Incorporate a review of algebra skills and coordinate geometry as indicated in the examples below.

Examples:
A. Find the value of \( x \) and list the length of the sides of \( \triangle ABC \) in order from shortest to longest if \( m\angle A = 5x - 5^\circ \), \( m\angle B = 4x^\circ \), and \( m\angle C = 17x + 3^\circ \).

Solution: \( x = 7 \), \( AC < BC < AB \)

B. \( \triangle DEF \) has vertices \( D(-2,1) \), \( E(5,3) \), and \( F(4,-3) \). List the angles in order from greatest to least.

Solution: \( \angle D, \angle E, \angle F \)

**Activity 12: The Triangle Inequality (GLE: 10)**

Students should work in groups of two or three. Give each group a set of straws which have been cut into different lengths. First, have students measure the length of each straw. Instruct students to make as many different triangles with the segments as possible within a certain time. Have them record all trials including sets that do not work. After the activity, have a whole class discussion about which combinations of triangles will form a triangle as opposed to those that would not form a triangle. Ask students to form conjectures on how they can determine if 3 given segments will form a triangle. Provide students with 4 to 5 real-life problems in which they would need to know which combinations of segments will form a triangle. Give students two side measures and ask them to determine the range of measures for the length of the third side.

Example: Two sides of a triangle measure 4 inches and 7 inches. What is the range of measures of the third side of the triangle?

Solution: \( 3 < x < 11 \).
Activity 13: Similar or Not? (GLEs: 10, 17, 19, 23)

Review with students the definition of similar figures. Use different activities which allow students to formalize the definition (i.e., corresponding angles are congruent and corresponding sides have the same ratios). For example:

- Have students construct a pair of triangles in which the angles are congruent, but the side lengths are not the same. Have students determine the ratios of the corresponding sides of the two triangles.

- Draw a triangle on a transparency and label the diagram with the measures of the angles and sides. Use an overhead projector to display the triangle’s image on the chalkboard or whiteboard. Have various students measure the angles and sides of the image and then determine the ratios of the corresponding sides. (This also works well as a teacher demonstration.)

Reinforce with students that constructing triangles with congruent angles (AA or AAA), creates similar, but not necessarily congruent, triangles. Lead a discussion about why congruent triangles are considered to be similar. Provide activities that allow students to investigate SSS and SAS similarity.

Once students have developed the definition of similar figures, provide students with diagrams that have pairs of similar triangles. Have students prove these triangles similar using the SSS similarity, SAS similarity, and AA similarity. Some pairs of triangles should also be congruent.

Activity 14: Conjectures about Quadrilaterals (GLE: 10)

Have students work in groups of two (preferred), three or four using an automatic drawer (such as found in The Geometer’s Sketchpad® software). The purpose of this activity is to allow students to investigate the properties of special convex quadrilaterals.

An example of the process to be used is provided below using a kite as the convex quadrilateral.

- Provide students with an electronic file in which a kite has been drawn. Have students measure the four angles and the four sides and record the measures.

- Instruct students resize the quadrilateral by dragging the vertices of the kite. Measure the angles and sides of the resized kite and record the information.

- Have students to resize and make measurements until they can form conjectures about the measures of the angles and the lengths of the sides in any kite (i.e., a kite has two pairs of congruent and adjacent sides; a kite has one pair of congruent angles which are formed by a pair of non-congruent sides).

- Instruct students to construct the diagonals of the kite and then answer questions relative to the behavior of the diagonals (e.g., Are diagonals perpendicular? Do the diagonals bisect the angles of the quadrilateral? Do the diagonals bisect each other?).
Repeat the process using trapezoids, isosceles trapezoids, parallelograms, squares, rectangles and rhombi. Have students compare the properties of the various convex quadrilaterals used in the investigation.

Lead a summary discussion of the conjectures and help students to organize the results by using classifications of quadrilaterals (e.g., any quadrilateral that is a parallelogram has congruent opposite angles and supplementary consecutive angles). Students should organize all of the properties and types of quadrilaterals in a graphic organizer like a Venn diagram.

**Activity 15: The Quadrilateral Family (GLEs: 10, 23)**

**Directions:**

Using a diagram similar to the one provided below, fill in the names of the quadrilaterals so that each of the following is used exactly once:

- PARALLELOGRAM
- SQUARE
- TRAPEZOID
- ISOSCELES TRAPEZOID
- KITE
- QUADRILATERAL
- RECTANGLE
- RHOMBUS

**Explanation:** Following the arrows: The properties of each figure are also properties of the figure that follows it.
- Reversing the arrows: Every figure is also the one that precedes it.
Have students complete the graphic organizer shown above and then lead a class discussion to summarize how different quadrilaterals are related to one another. Students should be able to identify a square as being a rectangle, rhombus, parallelogram, and quadrilateral and justify their reasoning.

**Sample Assessments**

**General Assessments:**

- The student will complete journal entries for this unit. The teacher will grade the journal. Journal topics could include:
  - Explain the statement “A square is a rectangle, but a rectangle is not a square.”
  - In an isosceles triangle, is a perpendicular bisector drawn from any vertex always the same segment as altitude and median? Explain your reasoning.
  - Suppose you have three different positive numbers arranged in order from greatest to least. Which sum is it most crucial to test to see if the numbers could be the lengths of the sides of a triangle? Explain your answer and use examples if necessary.
- The teacher will provide the student with a net which gives some of the angle measures and with other angles labeled with variables representing the measurements of the angles. The student will find all the missing angle measures using the angle relationships learned in the unit and defend his/her answers by identifying the property or properties used to determine each missing value. The nets should have special quadrilaterals and other polygons embedded within the diagram so that the properties learned must be used to find some of the angle measures.
- The student will write proofs of congruent or similar triangles using information provided by the teacher. The teacher will evaluate proofs for accuracy (use of correct postulates and theorems) and completeness (not missing any steps in the reasoning process), allowing the student to use any method of proof desired.

**Activity-Specific Assessments**

- **Activity 6**: The student will complete a product assessment in which he/she designs a 5 by 5 inch tile using various types of triangles. The triangle will be correctly marked to show an understanding of methods used to determine triangle congruency. Activity sheets are provided immediately following this section.
• **Activity 7:** The teacher will provide the student with different triangles and have him/her draw the angle bisectors, perpendicular bisectors, medians, and altitudes for the triangles. The teacher will provide one isosceles, one obtuse, one right, and one scalene triangle and have the student draw different special segments on each. The student will draw the altitude on the right and obtuse triangle, all three special segments on the isosceles triangle (one segment should satisfy this), and the angle bisectors on any type triangle. The student will also explain the processes used to make the drawings.

• **Activity 15:** The student will complete a Venn diagram to demonstrate understanding of the properties of the parallelograms discussed in class. See the example in the Resources. This example is only a guide and may be expanded to include include trapezoids, isosceles trapezoids, and kites by drawing a larger rectangle around the Venn Diagram shown and eliminating the requirement that properties numbers be shown (e.g., isosceles trapezoids have congruent diagonals and are not parallelograms so they would have to be drawn within a quadrilateral set and outside the parallelogram set. As a result, the congruent diagonal characteristic would need to be repeated.)
Instructions for Product Assessment
Activity 5

Your task is to design a tile in the 5-inch by 5-inch squares provided on the next two pages. There are two parts to this project.

Part I:
The drawings on your tile must meet certain specifications. You must have the following and you will be graded on the accuracy of the following.

1.) 2 congruent obtuse triangles which demonstrate congruency by ASA
2.) 2 congruent scalene triangles which demonstrate congruency by SSS
3.) 2 congruent isosceles right triangles which demonstrate congruency by SAS
4.) 2 congruent acute triangles which demonstrate congruency by AAS
5.) 1 equilateral triangle

You should have a minimum of 9 triangles in your design (i.e. your two acute triangles CANNOT double as your two scalene triangles). You may add other shapes once you are sure you have the required 9 triangles above.

On Part I, you must label and mark each pair of triangles according to one of the methods indicated in the directions above (see example below).

2 scalene triangles congruent by SSS
\[ \overline{AB} \cong \overline{DB} \]
\[ \overline{AC} \cong \overline{DC} \]
\[ \overline{BC} \cong \overline{BC} \]
\[ \triangle ABC \cong \triangle DBC \]

For Part II, you are to redraw your tile (without the markings and labels) in the square on the second page and COLOR it. Cut the tile out of the page and put your name, number and hour ON THE BACK! Do NOT glue it to another page, and do NOT staple it to part one. If you do not complete part two, the entire project will be returned to you, and you will lose one letter grade for each day late!!

DUE DATE:
### Part I

<table>
<thead>
<tr>
<th>Obtuse Triangles (ASA)</th>
<th>Scalene Triangles (SSS)</th>
<th>Isosceles Right Triangles (SAS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Acute Triangles (AAS)</th>
<th>Equilateral Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
<tr>
<td>□ □ □ □</td>
<td>□ □ □ □</td>
</tr>
</tbody>
</table>
Product Assessment Rubric

This is a checklist for evaluating your tile design. Your grade will be a percentage based on the number of requirements met.

**Are the following present?** 40%
1.) 2 congruent obtuse triangles [ ] yes [ ] no
2.) 2 congruent scalene triangles [ ] yes [ ] no
3.) 2 congruent isosceles right triangles [ ] yes [ ] no
4.) 2 congruent acute triangles [ ] yes [ ] no
5.) 1 equilateral triangle [ ] yes [ ] no

**Are the triangles marked by the correct method?** 40%
6.) ALL triangles labeled [ ] yes [ ] no
7.) Obtuse congruent by ASA [ ] yes [ ] no
8.) Scalene congruent by SSS [ ] yes [ ] no
9.) Isosceles right congruent by SAS [ ] yes [ ] no
10.) Acute congruent by AAS [ ] yes [ ] no

**Are the congruent triangles and parts listed correctly (based on markings)?** 20%
11.) Obtuse triangles and parts [ ] yes [ ] no
12.) Scalene triangles and parts [ ] yes [ ] no
13.) Isosceles right triangles and parts [ ] yes [ ] no
14.) Acute triangles and parts [ ] yes [ ] no
15.) Equilateral triangle [ ] yes [ ] no

**Following directions and promptness (for each “no” below you will lose one percentage point):**
16.) Tile drawn on handout and name on handout [ ] yes [ ] no
17.) Part two is colored [ ] yes [ ] no
18.) Part two is cut out and NOT attached by staple or glue [ ] yes [ ] no
19.) Name, number and hour on back of part 2 [ ] yes [ ] no
20.) Rubric turned in [ ] yes [ ] no
21.) Turned in on time [ ] yes [ ] no

Score \[4( ) + 4( ) + 2( )/10 = \]
Venn Diagram for Assessment for Activity 14

Directions: Label the Venn diagram below with the name and the number representing the properties for parallelograms. Remember, in a Venn Diagram each property should only be listed once.

1.) Diagonals are perpendicular.
2.) All four angles are right angles.
3.) Opposite angles are congruent.
4.) Diagonals bisect a pair of opposite angles.
5.) Diagonals are congruent.
6.) Opposite sides are congruent.
7.) Diagonals bisect each other.
8.) All four sides are congruent.
9.) Opposite sides are parallel.
10.) Consecutive angles are supplementary.
Answer Key

Parallelograms – 3, 6, 7, 9
Rhombii – 1, 4, 8
Squares
Rectangles – 2, 5
Geometry
Unit 5: Similarity and Trigonometry

Time Frame: Approximately four weeks

Unit Description

This unit addresses the measurement side of the similarity relationship which is extended to the Pythagorean theorem, its converse, and their applications. The three basic trigonometric relationships are defined and applied to right triangle situations.

Student Understandings

Students apply their knowledge of similar triangles to finding the missing measures of sides of similar triangles. This work is extended with use of the Pythagorean theorem to find the length of missing sides in a right triangle. The converse of the Pythagorean theorem is used to determine whether a given triangle is a right, acute, or obtuse triangle. Students can use sine, cosine, and tangent to find lengths of sides or measures of angles in right triangles and their relationship to similarity.

Guiding Questions

1. Can students use proportions to find the lengths of missing sides of similar triangles?
2. Can students use similar triangles and other properties to prove and apply the Pythagorean theorem and its converse?
3. Can students relate trigonometric ratio use to knowledge of similar triangles?
4. Can students use sine, cosine, and tangent to find the measures of missing sides or angle measures in a right triangle?
### Unit 5 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Simplify and determine the value of radical expressions (N-2-H) (N-7-H)</td>
</tr>
<tr>
<td>2.</td>
<td>Predict the effect of operations on real numbers (e.g., the quotient of a positive number divided by a positive number less than 1 is greater than the original dividend) (N-3-H)</td>
</tr>
<tr>
<td>3.</td>
<td>Define <em>sine</em>, <em>cosine</em>, and <em>tangent</em> in ratio form and calculate them using technology (N-6-H)</td>
</tr>
<tr>
<td>4.</td>
<td>Use ratios and proportional reasoning to solve a variety of real-life problems including similar figures and scale drawings (N-6-H) (M-4-H)</td>
</tr>
<tr>
<td><strong>Measurement</strong></td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>Model and use trigonometric ratios to solve problems involving right triangles (M-4-H) (N-6-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>10.</td>
<td>Form and test conjectures concerning geometric relationships including lines, angles, and polygons (i.e., triangles, quadrilaterals, and ( n )-gons), with and without technology (G-1-H) (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td>12.</td>
<td>Apply the Pythagorean theorem in both abstract and real-life settings (G-2-H)</td>
</tr>
<tr>
<td>17.</td>
<td>Compare and contrast inductive and deductive reasoning approaches to justify conjectures and solve problems (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td>18.</td>
<td>Determine angle measures and side lengths of right and similar triangles using trigonometric ratios and properties of similarity, including congruence (G-5-H) (M-4-H)</td>
</tr>
<tr>
<td>19.</td>
<td>Develop formal and informal proofs (e.g., Pythagorean theorem, flow charts, paragraphs) (G-6-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>23.</td>
<td>Draw and justify conclusions based on the use of logic (e.g., conditional statements, converse, inverse, contrapositive) (D-8-H) (G-6-H) (N-7-H)</td>
</tr>
</tbody>
</table>

### Sample Activities

**Activity 1: Striking Similarity (GLEs: 4, 10)**

Have students work in pairs. Each pair should receive a piece of grid paper with irregular shapes drawn on it similar to the ones below. Students should also be given a piece of grid paper which has larger squares but the sheet is the same size. For instance, the diagram below is an \( 8 \times 10 \) section of grid paper that might have 5 squares per inch. The teacher will give the students a blank \( 8 \times 10 \) grid that might be 4 squares per inch or 3 squares per inch. Students will then reproduce the shapes on the blank grid by drawing the segments in the corresponding squares on the blank grid. Once students have enlarged the figures, have students measure the segments and angles of both the original drawing and the new drawing. Have students participate in a discussion that describes the
relationship between pairs of corresponding angles and segments in the original and enlarged figures. Remind students about the information obtained in the previous unit about the corresponding sides and angles of similar triangles and have the students develop a definition for similar figures.

Activity 2: Similarity and Ratios (GLE: 4)

Instruct students to use equilateral triangle pattern blocks and cubes to make generalizations about the ratios of sides, areas, and volumes in similar figures using an activity like the one below.

- Given an equilateral triangle, create a similar triangle so the ratio of side lengths is 2:1. What is the ratio of areas of the two similar triangles? Now create a triangle similar to the original triangle so the ratio of side lengths is 3:1. What is the ratio of the areas of these two similar triangles?

  Sample sketches:

- Use other pattern block shapes to investigate other similar polygons in the same manner as described above and record your findings in the table below.

<table>
<thead>
<tr>
<th>description of similar shapes</th>
<th>ratio of sides</th>
<th>ratio of areas</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Based on your investigations in the two activities, make a generalization. If the ratio of sides of two similar polygons is \( n:1 \), what would the ratio of areas be?

- Given a cube, create a similar cube with ratio of edges 2:1 using cm or sugar cubes. What is the ratio of volumes? Create a similar cube with ratio of edges 3:1. What is the ratio of volumes? If the edges of two cubes were in a ratio of \( n:1 \), what would the ratio of volumes be? Record your findings in a table like the one below.

<table>
<thead>
<tr>
<th>Description of similar 3-D shapes</th>
<th>Ratio of edges</th>
<th>Ratio of volumes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Activity 3: Exploring Similarity Using Scale Drawings (GLEs: 2, 4, 10)**

Visit the website [http://www.eduref.org/cgi-bin/printlessons.cgi/Virtual/Lessons/Mathematics/Geometry/GEO0003.html](http://www.eduref.org/cgi-bin/printlessons.cgi/Virtual/Lessons/Mathematics/Geometry/GEO0003.html) to access the information for this activity. Which allows students to use their knowledge of similar figures. Have students create a scale model of a rectangular box, measure the original and the scale model, and calculate the surface area and volume of the boxes. In addition to the procedures listed on the website, before they actually do the calculations, have students predict what they think the surface area and volume should be based on the measurements of the two figures. This will assist students in determining if their solutions are reasonable and allows them to apply the information learned in Activity 2. There should be no need for students to access the website since teachers can print the material for the class.

Since students are measuring the items themselves, help them to understand why their results may not be exactly what theory says they should be.

**Activity 4: Spotlight on Similarity (GLEs: 2, 4, 19, 23)**

Students investigate the following problem:
- A spotlight at point P throws out a beam of light.
- The light shines on a screen that can be moved closer to or farther from the light. The screen at position A is a distance A from the light and at position B is a distance B.
- The lengths a and b indicate the lengths of the light patch on the screen.
• Show that the ratio of the length $b$ to the length $a$ depends only on the distances $A$ and $B$, and not on the angle $y$ of the beam to the perpendicular nor on the angle $x$ of the beam itself.

Using similar triangle relationships, \( \frac{a+c}{A} = \frac{b+d}{B} \). This means \( \frac{a}{A} + \frac{c}{A} = \frac{b}{B} + \frac{d}{B} \) and \( \frac{c}{A} = \frac{d}{B} \) because the triangles are similar. Hence, \( \frac{a}{A} = \frac{b}{B} \), which is equivalent to \( \frac{b}{a} = \frac{B}{A} \). The ratio \( \frac{B}{A} \) is independent of the angles $x$ and $y$ and is the scale factor relating the distances of the two screens and the sizes of the images on the two screens. Students should be able to predict whether the scale factor causes one of the triangles to get larger or smaller.

Once students develop an understanding of the term scale factor, give students the measurements of certain figures and a scale factor. Have them predict whether the new figure is going to be larger or smaller than the given measures and then check themselves by finding the actual measures by using proportions.

**Activity 5: Applying Similar Figures (GLEs: 2, 4, 18)**

Give students various real-life situations in which similar triangles are used to find missing measures (i.e. shadow problems, distance across a river, width of a lake). The types of triangles should vary. Have students discuss why the triangles are similar before finding the requested missing measures. Provide students with practice in determining the missing sides of other pairs of similar figures in real-life settings.
Example:
Alex is having a snapshot of his grandparents enlarged. The original snapshot is 4 inches by 6 inches. He needs the enlarged photo to be at least 13 inches on the shortest side. What must the minimum length be of the longer side? Solution: 19.5 in.

Activity 6: Parts of Similar Triangles (GLEs: 2, 4, 10)
Students should investigate how the lengths of the special segments in similar triangles relate to the measures of the sides of the similar triangles. Have them construct similar triangles, either with a drawing program or by hand, and draw their altitudes, angle bisectors, and medians. Instruct students to determine the scale factors of the sides and compare them to the ratios of the special segments. Ask students to investigate the ratio of the perimeters of the similar triangles. Lead a class discussion to summarize that the ratios of the sides, altitudes, medians, angle bisectors, medians, and perimeters in similar figures are equal.

Activity 7: Midsegment Theorem for Triangles (GLEs: 4, 10, 18)
Separate the class into groups of four. Give each group a sheet of tracing paper (or patty paper) and have them draw a triangle of any type and cut it out. Have students:
1. Find the midpoints of any two sides of the triangle by folding.
2. Fold (or draw) the segment that connects the two midpoints. Tell students that this is the midsegment and have them define the term based on what they have done so far.
3. Unfold the triangle and make any observations that look true about the triangle and the midsegment.
4. Fold and unfold the remaining two midsegments of the triangle.
5. Have students make observations regarding the three midsegments of the triangle. Ask them to look for a geometrical relationship between the midsegments and the sides and to determine the numerical relationship between the lengths of the midsegments and the lengths of the sides (i.e., each midsegment is parallel to one side of the triangle and has a length that is one-half the length of that side).
If there is access to a computer drawing program such as Geometer’s Sketchpad®, use it to construct the midsegments of several other triangles to determine if the observations made in the part above hold true.

Lead the class in a discussion which includes the use of similar triangles to prove the Midsegment Theorem (the segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half its length). Ask students to discuss the relationship of the triangle formed by the three midsegments to the original triangle (i.e., the inner triangle is similar to the original and its perimeter is half the perimeter of the original triangle).
Activity 8: Pythagorean Theorem (GLE: 12)

Provide students with a pair of similar right triangles whose leg measures are known. Ask students to determine if the triangles are similar and, if so, to provide a proof (i.e., the right angles are congruent and the legs in the two triangles are proportional). Have students calculate the length of the hypotenuse of each triangle. Ask: What is the scale factor between the two similar triangles? Is the hypotenuse of one triangle a multiple of the hypotenuse of the second triangle? What is the multiple? Students should recognize and use common Pythagorean triples (e.g., 3-4-5, 5-12-13, 7-24-25, 8-15-17) and their multiples as shortcuts to solving problems. For example, if a right triangle has lengths of 15, 36, 39, the missing side is 36 since 15, 36, 39 is three times 5-12-13.

Activity 9: Proving the Pythagorean Theorem and Its Converse (GLEs: 17, 19, 23)

Have students prove the Pythagorean theorem and its converse. To prove the Pythagorean theorem, have students use manipulatives or patty paper to construct squares with side lengths of \(a\), \(b\), and \(c\) to show that \(a^2 + b^2 = c^2\). Have students test the converse of the Pythagorean theorem by constructing a triangle using the same three lengths \(a\), \(b\), and \(c\). Lead a class discussion in which students indicate that the SSS triangle congruence postulate verifies that all triangle of lengths \(a\), \(b\), and \(c\) are congruent and the triangle constructed must be congruent to the original triangle.

Activity 10: Application of the Converse of the Pythagorean Theorem (GLEs: 10, 12)

In this activity students should apply the converse of the Pythagorean Theorem to determine if a triangle is right, acute, or obtuse. Give the students several different sets of measures that form a triangle (be sure that most of them are NOT right triangles). Have students apply the converse of the Pythagorean Theorem to determine which of the trios forms a right triangle. Using a computer drawing program like Geometer’s Sketchpad, have students construct triangles using side lengths that do not form right triangles to determine that some triangles are acute while others are obtuse. Have students make a conjecture about the sum of the squares of the smaller sides in relation to the square of the largest side in acute and obtuse triangles. Students should explain that if \(a^2 + b^2 < c^2\) then the triangle is obtuse and if \(a^2 + b^2 > c^2\), then the triangle is acute. Ask students to classify other triangles based only on the lengths of their sides.

If a drawing program is not available, provide students with diagrams in which the triangles have been drawn to scale and the lengths of sides are labeled. Have students apply the Pythagorean theorem (or the rules concerning Pythagorean triples) to determine which triangles are right triangles. For the remaining triangles, have students use a protractor to measure angles, classify each triangle as acute or obtuse, and then determine the relationship between \(a^2 + b^2\) and \(c^2\) in the two types of triangles.
Activity 11: Trigonometry (Using Technology) (GLEs: 3, 8, 12)

The website, http://catcode.com/trig/index.html, provides a series of activities that define and help explain the uses of trigonometry. The activities help students to expand their understandings of similar figures as they apply to the study of trigonometry. The first five demonstrations are appropriate as introductory activities.

Activity 12: Special Right Triangles (GLEs: 1, 3, 10, 12, 18)

Have students explore special right triangles by starting with an equilateral triangle with side lengths of 2 units. Have students construct an altitude to create two 30°-60°-90° triangles. Identify the parts of the 30°-60°-90° triangle as short leg, long leg, and hypotenuse. The resulting right triangles have a short leg of 1 unit. A 30°-60°-90° triangle whose short leg is 1 is called the unit triangle. Have students use the Pythagorean theorem to calculate the length of the long leg (side opposite 60°) in simplified radical form. Next, have students create a unit triangle for 45°-45°-90°, using 1 unit as the length of each of the two legs. Using the Pythagorean theorem, students will calculate the length of the hypotenuse in simplified radical form.

Repeat the activity several times but use different measures for the sides of the equilateral triangle (e.g., start with an equilateral triangle whose sides are 4 units, 6 units). Do this several times until students see a pattern in the numbers. Write these as formulas: $\text{short leg} = \frac{1}{2} \times \text{hypotenuse}$ and $\text{long leg} = \sqrt{3} \times \text{short leg}$ in 30°-60°-90° triangles. For 45°-45°-90° triangles, the relationship is $\text{hypotenuse} = \sqrt{2} \times \text{leg}$. Additionally, show students how proportions are an alternative way of calculating the same values.

To help students become familiar with the definition of sine and cosine, have them calculate the ratios using the side lengths of special right triangles. Have students use the examples from the above activity and the definition of sine to determine that $\sin (30°) = \frac{1}{2}$. Allow them to use the calculator’s sin function to verify this. (This step may require instruction on the use of the calculator.) Help students to understand that formulas, proportions, and the trig functions are related to each other and that each is a different way to write the ratios that exist. See that students become familiar with the idea that trigonometric functions represent ratios of sides in a right triangle.

Activity 13: Trigonometry (GLEs: 2, 3, 8, 12, 18)

Extend Activity 12 to define the cosine and tangent ratios in right triangles. Be sure to include information which shows students how to find the measures of the acute angles in a right triangle if the side lengths are known. Assist students in learning to use the calculator to find trig ratios and to use the ratios to solve problems. In order to facilitate understanding, have students read information from a standard trig table. For example, in
order to solve \( \tan x = \frac{12}{17} \), students need to understand that there is one angle which has the same decimal ratio as 12 divided by 17. Looking through the list of tangent ratios to find this number helps students understand that the calculator has these ratios stored in its memory. When the student requests \( \tan^{-1} \left( \frac{12}{17} \right) \), he/she is requesting the calculator to search for the angle whose ratio is the same as 12 divided by 17.

Have students practice finding the measures of missing sides and angles by applying the trigonometric ratios to right triangles. Once an understanding of the process is mastered, have students apply the trigonometric ratios to real-life problems. These problems can be to find the differences between the heights of two buildings, distance two boats are apart from each other, construction of airplanes, angles of elevation or depression, etc.

**Sample Assessments**

**General Assessments**

- The student will create a portfolio containing samples of his/her activities. For instance, the student could choose a particular drawing from class and enlarge it using a given scale. In this entry he/she would also explain the process and how to prove that the new drawing is similar to the given drawing.
- The student will complete journal entries for this unit. For example:
  - Discuss the proof for the special right triangles: 30°-60°-90° and 45°-45°-90°. In your discussion, explain why this information can be generalized to all triangles that have these angle measures.
  - Explain how the Pythagorean theorem can be used to determine if a triangle is a right, obtuse, or acute triangle.
- The student will find pictures of similar figures in magazines, newspapers, or other publications and explain how he/she knows that the figures are similar. The teacher will challenge the student to find pairs of similar figures that are not congruent.

**Activity-Specific Assessments:**

- Activity 1: The teacher will give the student a floor plan for a house. The floor plan should not have any measurements on it. The student will enlarge the floor plan to the size of a poster using a given scale. The student will find the actual dimensions of the rooms and the dimensions of the entire house by from scale used to create the floor plan.

- Activity 5: The teacher will provide instructions for making and using a hypsometer. These instructions are provided at the end of this unit. The
student will write a rational for the proportion that is given in the instructions. The student will determine the height of various objects throughout school showing all calculations necessary to indirectly find the height of the chosen object.

- **Activity 11:** The student will use a clinometer to determine the height of something on the grounds of the school (e.g., flag pole, light post, goal post) using the trigonometric functions. The student will produce a scaled diagram of the measurements made and show all calculations used to indirectly calculate the height of the chosen object using trig functions. Instructions for making a clinometer can be found in most geometry textbook and on numerous websites such as [http://www.zephyrus.demon.co.uk/geography/resources/fieldwork/fluvial/grad.html](http://www.zephyrus.demon.co.uk/geography/resources/fieldwork/fluvial/grad.html).
Making and Using a Hypsometer

**Format:** Individual or Small Group

**Objectives:** Participants use the hypsometer and their knowledge of the proportional relationship between similar triangles to determine the height of an object not readily measured directly.

**Materials:** For each hypsometer you need a straw, decimal graph paper, cardboard, thread, a small weight, tape, a hole punch, scissors, and a meter stick.

**Time Required:** Approximately 90 minutes

**Directions:**

To make the hypsometer:

1. Tape a sheet of decimal graph paper to a piece of cardboard.

2. Tape the straw to the cardboard so that it is parallel to the top of the graph paper.

3. Punch a hole in the upper right corner of the grid. Pass one end of the thread through the hole and tape it to the back of the cardboard. Tie the weight to the other end of the thread.
To use the hypsometer:

4) Use a meter stick to measure the height of your eye and the distance to the object you wish to measure.

5) Look through the straw at the top of the object you wish to measure. Use your finger to hold the string where it hangs so you can record the hypsometer reading. Be careful not to move your finger when you take the hypsometer from your eye to read the measurement!

6) To find the height of the flagpole, recognize that triangles ABC and DEF are similar. Thus, BC can be found using the following ratio:

\[
\frac{AC}{DF} = \frac{BC}{EF}
\]

Reference: NCTM Addenda Series, *Measurement in the Middle Grades*
Geometry
Unit 6: Area, Polyhedra, Surface Area, and Volume

Time Frame: Approximately five weeks

Unit Description

This unit provides an examination of properties of measurement in geometry. While students are familiar with the area, surface area, and volume formulas from previous work, this unit provides justifications and extensions of students’ previous work. Significant emphasis is given to 3-dimensional figures and their decomposition for surface area and volume considerations.

Student Understandings

Students understand that measurement is a choice of unit, an application of that unit (covering, filling) to the object to be measured, a counting of the units, and a reporting of the measurement. Students should have a solid understanding of polygons and polyhedra, what it means to be regular, what parallel and perpendicular mean in 3-dimensional space, and why pyramids and cones have a factor of $\frac{1}{3}$ in their formulas.

Guiding Questions

1. Can students find the perimeters and areas of triangles, standard quadrilaterals, and regular polygons, as well as irregular figures for which sufficient information is provided?
2. Can students provide arguments for the validity of the standard planar area formulas?
3. Can students define and provide justifications for polygonal and polyhedral relationships involving parallel bases and perpendicular altitudes and the overall general $V = Bh$ formula, where $B$ is the area of the base?
4. Can students use the surface area and volume formulas for rectangular solids, prisms, pyramids, and cones?
5. Can students find distances in 3-dimensional space for rectangular solids using generalizations of the Pythagorean theorem?
6. Can students use area models to substantiate the calculations for conditional/geometric probability arguments?
Unit 6 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Measurement</strong></td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>Find the volume and surface area of pyramids, spheres, and cones (M-3-H) (M-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>9.</td>
<td>Construct 2- and 3-dimensional figures when given the name, description, or attributes, with and without technology (G-1-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Form and test conjectures concerning geometric relationships including lines, angles, and polygons (i.e., triangles, quadrilaterals, and (n)-gons), with and without technology (G-1-H) (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td>12.</td>
<td>Apply the Pythagorean theorem in both abstract and real-life settings (G-2-H)</td>
</tr>
<tr>
<td>18.</td>
<td>Determine angle measures and side lengths of right and similar triangles using trigonometric ratios and properties of similarity, including congruence (G-5-H) (M-4-H)</td>
</tr>
<tr>
<td>19.</td>
<td>Develop formal and informal proofs (e.g., Pythagorean theorem, flow charts, paragraphs) (G-6-H)</td>
</tr>
<tr>
<td>21.</td>
<td>Determine the probability of conditional and multiple events, including mutually and non-mutually exclusive events (D-4-H) (D-5-H)</td>
</tr>
</tbody>
</table>

**Sample Activities**

**Activity 1: Why Does That Formula Work? (GLE: 10)**

*Teacher Note: Students should be able to find the area of basic figures such as rectangles, triangles, and trapezoids through work in previous grades. This activity explores how the formulas were derived and introduces a formula for the area of a rhombus.*

Have students investigate why the rectangle area formula is base times height. The easiest way to have students see this is to use grid paper and count the squares. Have them investigate a parallelogram that is not a rectangle or a square and recognize that the formula for the area of a rectangle is the same formula as the area of a parallelogram. Have students review the definition of distance between two lines to understand why the height of a general parallelogram is different than the measure of the sides. Emphasize that although students have always learned that the area of a rectangle is length times width, the better interpretation is base and height since a rectangle is special type of parallelogram. This makes the formula applicable to any quadrilateral that is a parallelogram. For most students this will be only a matter of changing the variables from prior knowledge.
After investigating the parallelogram, have students examine the formulas for other plane figures—triangles, trapezoids, and rhombi. Have students explain how each formula is derived from other formulas as well as how to apply them to various problem situations.

Review with students the fact that the diagonals of a rhombus are perpendicular and bisect each other. Have students generate the formula for finding the area of a rhombus using the lengths of the diagonals \( A = \frac{1}{2}d_1d_2 \). Lead a discussion in which students have the opportunity to consider other quadrilaterals whose areas could be found using the same formula (i.e., squares and kites). This discussion should be based upon the characteristics of the diagonals that were investigated in the unit on quadrilaterals.

**Activity 2: Area of Regular Polygons (GLEs: 7, 9, 10, 12, 18)**

*Teacher Note: While GLE 7 refers to the volume and surface area, the area of regular polygons should be reviewed to assist students in finding the volume and surface area of polyhedra with bases that are regular polygons.*

In this activity students use hinged mirrors, protractors, and rulers to draw regular polygons and investigate the measures of their central angles. This will be pivotal to helping find the lengths of apothems that are not given when finding the area of regular polygons. A similar activity with activity sheets are available at [http://illuminations.nctm.org/index_d.aspx?id=379](http://illuminations.nctm.org/index_d.aspx?id=379).

1. Have students draw a line on a plain sheet of paper. Position the hinged mirror so that the sides of the mirror intersect the line at two points that are equal distances from the hinge of the mirror.

2. Remind students to sketch what they see and then place the protractor on top of the mirror to determine the angle of the mirror. Record this measurement; this is the measure of a central angle of the polygon that they see in the mirror.

3. Have the students open the mirror wider; students must make sure that the mirror intersects the line at equal distances from the hinge. They should observe what happens to the figure, sketch the new figure and record the new angle measurement.

4. Have students open or close the mirror until a regular hexagon is formed and record the angle of the mirror. Then students should answer the following questions about the six-sided figure:
a. What is the sum of the angle measures for those angles whose vertex is the mirror’s hinge.
b. How many angles are there in the image for each number of sides and what is the measure of each of those angles?
c. How does the measure of the hinged mirror compare to the measure you calculated in b above?

5. Have students use the hinged mirror to find the central angle for the following regular polygons: triangle, square, pentagon, octagon, decagon, and dodecagon.

Direct students to construct a circle using a compass and inscribe a regular hexagon in the circle by making 6 congruent arcs along the circle. Have students divide the hexagon into six congruent triangles and write the formula for the area of the hexagon. Lead a class discussion to determine that students were able to generate the formula as $A = 6\left(\frac{1}{2}sa\right)$ where $s$ is the measure of one side of the hexagon, and $a$ is the height of the triangle, or the apothem of the hexagon. Ask students to consider the meaning of $6s$ as this expression relates to the hexagon. Once it has been established that $6s$ is the perimeter of the hexagon, rewrite the formula for the area of a hexagon as $A = \frac{1}{2}Pa$ where $A$ is the area of the polygon, $P$ is the perimeter of the polygon, and $a$ is the apothem of the polygon. This formula can be used for all regular polygons.

Have students practice calculating the area of other regular polygons. Provide diagrams with labeled measurements or provide real objects which students can measure the parts needed for the formula. Provide instances in which students must use special right triangles or trigonometric ratios to find the apothem or the side measure of the polygon.

**Activity 3: Experiment with Volume (GLE: 7, 9, 10)**

*Teacher Note: This activity relates to the construction of 3-D figures as stated in GLE 9; however, it is also included as a review of the volume concepts mastered in grade 8.*

**The Problem:** Take a sheet of paper and roll it up to form a baseless cylinder. Now, take another sheet, rotate the paper, and form another baseless cylinder. Think about the volume of each cylinder and make a prediction.
The Prediction

- Would the two volumes be equal?
- Would the short cylinder have greater volume?
- Would the tall cylinder have greater volume?

Explanation: Why did you predict as you did?

Demonstration:

Tape two sheets of paper to form the two cylinders, one short and one tall. (Stiff paper is helpful. Transparency sheets may be used). Hold the tall cylinder upright in a shallow box and fill with rice. Now fill the shorter cylinder, and compare the two amounts of rice. Was your prediction correct?

Calculation: Calculate both volumes.

\[
V = \pi r^2 h
\]

\[
V = \pi r^2 h
\]

Lead the students in a discussion as to why the volumes are not the same even though the numbers are the same. Include in the discussion a study of the areas of the bases and a determination of the effects of squaring the radius when calculating the areas of those bases.

Activity 4: Cube Coloring Problem (GLEs: 9, 10)

Teacher Note: This activity provides a review of surface area and volume of prisms mastered in grade 8.
Overview:

Investigate what happens when different sized cubes are constructed from unit cubes, the surface areas are painted, and the large cubes are taken apart. How many of the 1×1×1 unit cubes are painted on three faces, two faces, one face, no face?

Objective: Students will be able to

- work in groups to solve a problem.
- determine a pattern from the problem.
- predict the pattern for larger cubes.
- graph the growth patterns.
- express the pattern algebraically.

Resources/Materials: Students will need a large quantity of unit cubes (sugar cubes can be used), graph paper, and colored pencils or markers.

Activities and Procedures:

- Hold up a unit cube. Tell students this is a cube on its first birthday. Ask students to describe the cube (eight corners, six faces, twelve edges). Find the cube’s total surface area and volume.
- Ask student groups to build a “cube” on its second birthday, that is, to double the length of each side of the cube. Ask the students to describe it in writing. Find the surface area and volume.
- Ask students how many unit cubes it will take to build a cube on its third birthday, fourth, fifth . . . . Find the surface area and volume on each of the cube’s birthdays.
- Pose this coloring problem: The cube is ten years old. It is dipped into a bucket of paint. After it dries, the ten-year-old cube is taken apart into the unit cubes. How many faces are painted on three faces, two faces, one face, no face? (If using sugar cubes, students can make a model and place a dot on each exposed side with a marker.)
- Have students chart their findings, including surface area and volume, for each age cube up to ten and look for patterns.
- Have students use exponents to write the number of unit cubes needed to make a larger cube. Expand this to the number of cubes painted on three faces, two faces, one face, or no face.
- Have students graph the findings for each dimension of cube up to ten and look for the graph patterns.

Tying It All Together:

This activity provides students with the chance to estimate, explore, use manipulatives, predict, and explain orally and in writing. Students will note that the three painted faces are always the corners—eight on a cube. The cubes with two faces painted occur on the
edges between the corner and increase by twelve each time. The cubes with one painted face occur as squares on the six faces of the original cube. The cubes with no painted faces are the cubes within the cube. Have students explain where the formulas for surface area and volume of a cube come from after this activity.

This is an excellent way for students to become involved in exploring a problem of cubic growth.

**Activity 5: Cylinder in 3-D (GLEs: 9, 10, 19)**

*Teacher Note: This activity also provides a review of a surface area formula mastered in grade 8.*

Use groups of two.

**Materials:**

- compass
- scissors
- tape
- metric ruler

Draw the figure shown below on a sheet of paper or thin cardboard, using the indicated measurements. Cut out this net and then fold and tape the edges to form a cylinder.

![Diagram of a cylinder net](image)

Generalize the dimensions of the cylinder and instruct students to generate the general formula for finding the surface area and volume of a cylinder.

**Activity 6: Building a Pyramid (GLEs: 9, 10, 12)**

Have students work in pairs and use the materials provided to construct a square pyramid which has a height of 7 cm and a base length of 4 cm. Each of the two students is to make a pyramid, but the use of pairs allows them to think through the process together. Remind students that they should use the Pythagorean theorem to figure the slant height (height of the triangle for one of the sides) in order to construct the faces of the pyramid.
Materials:
- two \(8\frac{1}{2} \times 11\) sheets of paper
- two pairs of scissors
- two rulers
- tape

The most expedient way to build the pyramid is to make a net consisting of a square with an isosceles triangle drawn on each side. The isosceles triangle should be constructed so that the height of the triangle is the calculated length of the pyramid’s slant height. Students who have trouble visualizing the net may use other methods, such as drawing each side individually and taping them all together. The teacher may want to make two paper models in advance of the activity and then dismantle one of the pyramids to show what the net looks like. If necessary, lead a discussion of the location of each measurement on the net in comparison to the 3-dimensional pyramid.

Have students discuss ways they might find the volume of the pyramid. Also lead the students in a discussion about the information that would be necessary to find the unknown height of a pyramid (or possibly an unknown base length of the pyramid if the height were given). Ask students to apply this knowledge by creating another pyramid with a different regular polygon as the base.

**Activity 7: Surface Area (GLE: 7)**

Using their models from Activity 6, have students determine the total surface areas of their constructed pyramids and describe the process of finding the surface area and volume of the pyramids.

**Activity 8: Volumes of Pyramids and Cones (GLE: 7)**

Compare the volumes of a pyramid and prism with the same base and height as a demonstration using volume model kit. (If enough model kits are available, have students work in groups of 2 or 3 when completing the activity. Students could also make their own models using old manila file folders and then complete the activity.)

Fill the pyramid from the kit with rice or unpopped popcorn. Ask students to estimate how many times the pyramid must be filled in order for the prism to be filled. Do the same thing with a cone and cylinder. Develop the concept that the volume of a cone (pyramid) is one-third the volume of a cylinder (prism) if the two solids have the congruent heights and bases. As an extension, ask students to estimate the relationship between the cone and the sphere which is also a part of the kit. Since the cone must be filled twice before the sphere is filled, the sphere is twice as large as the cone’s volume or...
the volume of the cylinder. Provide real-life applications in which students must find the volumes of cones, pyramids, prisms, and cones.

**Activity 9: Prove It! (GLEs: 7, 19)**

Have students construct a cone with a specified base radius and a specified height. See that students use the Pythagorean theorem to find the radius of the circle from which to cut the lateral surface of the cone. They will also find that the circumference of the base is used to determine what percent of the cone’s circle (used to make the lateral surface of the cone) is needed when forming the cone. Once this is done, generate the formula for finding the lateral surface area of a cone.

**Activity 10: More with Volume and Surface Area (GLE: 7)**

Have students review the processes for finding the surface area of prisms and pyramids. They should generalize that to find the total surface area of a prism or pyramid, they need to find the sum of all the areas of the lateral faces and the base(s). The volume of the prisms can be generalized as the product of the area of the base and the height of the prism. The volume of the pyramids should be generalized as finding \( \frac{1}{3} \) of the product of the area of the base and the height of the pyramid.

After these generalizations are made, have students practice finding the surface area and volume of prisms and pyramids with regular polygons as their bases.

**Activity 11: Volume of Irregular Objects (GLE: 7)**

*Teacher Note: Although GLE 7 refers to only the volume of pyramids, cones, and spheres, this activity gives students another opportunity to determine the volume of an object of shapes for which no formula exists.*

Provide students with a cylindrical object whose volume can be calculated and with markings to measure a predetermined amount of water (a beaker from a science class would do well). Ask them to place water in the beaker but not to fill it to the top. Discuss with the students the volume of water in the beaker. Ask them to place an irregular object, like an egg-shaped paperweight, into the water, being careful not to spill any water. Note the displacement of the water and determine the volume of the paperweight.

Next, repeat the activity with a tube of toothpaste. Have the box for the tube on hand as well. After students determine the volume of the tube of toothpaste, have them determine the volume of the box. Instruct students to determine what percent of the box is used by the toothpaste and what percent is empty space. Use this activity for a discussion about packaging efficiency.
Activity 12: Geometric Probability (GLE 21)

Students should be given problems that require them to find the area of a variety of shapes. This should include basic figures as well as figures within other figures, or combinations of figures. Ask students to find the probability of randomly selecting a point in a shaded region of the given figure.

Example:
Mark created a game consisting of 32 squares on a rectangular game board. The board measures 1-foot by 2-feet. 16 of the squares are 3-inches by 3-inches while the other 16 squares are 2-inches by 2-inches. He earns 3 points for hitting the board and not hitting a square, 5 points for hitting one of the larger squares, 10 points for hitting one of the smaller squares. What is the probability that he will earn 10 points with one throw of a dart? Solution: \( \frac{9}{288} = \frac{1}{32} = 22\% \).

Ask students to apply geometric probability using the length probability postulate—the probability of a point lying on a smaller portion of a segment is equal to the length of the smaller portion divided by the length of the entire segment.

Example:
A radio station will play the song of the day once during each hour. The 101\textsuperscript{st} caller will win $100. If you turn on the radio at 2:35 p.m., what is the probability that you have missed the start of the song during the 2:00 p.m. to 3:00 p.m. hour? Solution: \( \frac{35}{760} = \frac{7}{12} = 58\% \).

Sample Assessments

Performance and other types of assessments can be used to ascertain student achievement. Examples include:

General Assessments:

- The student will complete journal entries for this unit. Suggested topics include:
  - Explain why pyramids and cones have \( \frac{1}{3} \) as a factor in their formulas for volume.
  - Show how to find the volume and surface area of a solid that combines a cylinder with a cone, or a prism with a pyramid. Be specific.
- The teacher will provide the student with three-dimensional models. The student will sketch diagrams and take appropriate measurements from actual objects needed to calculate volume and surface area. The student will label sketches with measurements and then show the process used to calculate volume and surface area. Since this task is time consuming, the student will be given no more than three objects, some type of prism, either a cone or pyramid, and a cylinder.
Activity-Specific Assessments:

- **Activity 1**: The student will determine the total living area and total area of a given floor plan. This floor plan should have odd-shaped rooms that would require using most of the formulas discussed in this activity.

- **Activity 6**: The student will build a pyramid with a minimum surface area and minimum volume. The student will show the measurements of the bases, height of the faces, and the height of the pyramid based on the given minimum surface area and volume.

- **Activity 11**: The student will design a container to hold a specific volume of a specified product. The teacher will assign each student a different volume and specific shape or assign the volume and allow the student to choose the shape. The student will design a label for the container and determine the area of that label. The student will create a newspaper advertisement about the product to fit within a specified area.
Geometry
Unit 7: Circles and Spheres

Time Frame: Approximately five weeks

Unit Description

This unit focuses on justifications for circular measurement relationships in two and three dimensions, as well as the relationships dealing with measures of arcs, chords, secants, and tangents related to a circle. It also provides a review of formulas for determining the circumference and area of circles.

Student Understandings

Students can find the surface area and volume of spheres. Students understand the relationship of the measures of minor and major arcs to the measures of central angles and inscribed angles, and to the circumference. They also understand the relevance of tangents in real-life situations and the power of a point relationship for intersecting chords.

Guiding Questions

1. Can students provide an argument for the value of $\pi$ and the way in which it can be approximated by polygons?
2. Can students provide convincing arguments for the surface area and volume formulas for spheres?
3. Can students apply the circumference, surface area, and volume formulas for circles, cylinders, cones, and spheres?
4. Can students apply geometric probability concepts using circular area models and using area of a sector?
5. Can students find the measures of inscribed and central angles in circles, as well as measures of sectors, chords, and tangents to a circle from an external point?
6. Can students use the power of a point theorem (intersecting chords and intersecting secants) to determine measures of intersecting chords in a circle?
Unit 7 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Measurement</strong></td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>Find volume and surface area of pyramids, spheres, and cones (M-3-H) (M-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>13.</td>
<td>Solve problems and determine measurements involving chords, radii, arcs, angles, secants, and tangents of a circle (G-2-H)</td>
</tr>
<tr>
<td>17.</td>
<td>Compare and contrast inductive and deductive reasoning approaches to justify conjectures and solve problems (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td>19.</td>
<td>Develop formal and informal proofs (e.g., Pythagorean theorem, flow charts, paragraphs) (G-6-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>21.</td>
<td>Determine the probability of conditional and multiple events, including mutually and non-mutually exclusive events (D-4-H) (D-5-H)</td>
</tr>
<tr>
<td>22.</td>
<td>Interpret and summarize a set of experimental data presented in a table, bar graph, line graph, scatter plot, matrix, or circle graph (D-7-H)</td>
</tr>
</tbody>
</table>

**Sample Activities**

**Activity 1: Discovering π Worksheet (GLEs: 13, 17)**

Use the simulation of throwing a dart at a circle inscribed in a square found at the website, [http://www.mste.uiuc.edu/activity/estpi/](http://www.mste.uiuc.edu/activity/estpi/). Discuss how the area of the figures relates to the value of \(\pi\).

**Activity 2: Derivation of the Area of a Circle Formula (GLEs: 13, 17)**

Lead students in an exercise to show how the formula for the area of a circle can be developed. Have students cut a circle into 8 or 16 sectors, rearrange the sectors to form a parallelogram, and then use algebra to generate the area formula from the formula for the area of the parallelogram. Using the process allows students to review the circumference formula.

Another derivation is to increase the number of sides of a regular polygon. The apothem of the regular polygon becomes the radius of a circle that is generated as the number of sides in the polygon increases. Have students derive the area of a circle formula by looking at the process for calculating the area of a regular polygon (i.e., area of one triangle times number of triangles) and generalize this formula as the length of the polygon’s sides gets smaller and smaller. Using this technique allows students to review the process and/or formula for finding the area of a regular polygon.
Have a class discussion about the use of deductive logic in this process. What assumptions must be accepted as truth? How is the process different from induction?

**Activity 3: Throw That Dart! (GLEs: 13, 21)**

Provide students with several “dart” boards made of circles. For example, use circles on squares with the circles cut up into quarter pieces and placed in the corners of a square, or use several concentric circles. Have students shade in some of the circular regions, determine the areas of these regions, and then figure the probability of a dart which is thrown randomly at a dartboard landing in a shaded region (assume each dart hits the board). Have students create dartboards that possess specific probabilities for a randomly thrown dart landing in a shaded region.

**Activity 4: Central Angles and Arcs (GLE: 13)**

Provide pairs of students with a diagram containing a circle with a given radius length and with central angles labeled 1, 2, and 3. The measures of the central angles should be in the ratio of 2:3:4. Have students determine the measures of the central angles and lengths of the respective arcs. Repeat this activity for other circles and other ratios.

Review with students the fact that the sum of the central angles of a circle is 360°. If needed, have students use string or a tailor’s tape measure to find the circumference of the circle and to determine the length of one of its arc. Help students to internalize the concept that the ratio of the arc’s measure to 360 degrees is the same ratio as the arc’s length to the circle’s circumference by repeating this activity many times.

Provide opportunities for students to find both arc measure and arc length of major and minor arcs using the formula \[
\frac{\text{arc measure}}{360^\circ} = \frac{\text{arc length}}{\text{circumference}}.
\]

**Activity 5: Concentric Circles (GLE: 13)**

Provide students with several diagrams of concentric circles and various central angles. Be sure to have the central angles labeled and each point where the radii intersect each of the circles labeled. Have students find the measure of each of the central angles using a protractor. Make certain that students understand that the measure of an arc is the same as the measure of its central angle. Have students measure the radii of each of concentric circles and calculate the length of each arc intercepted by the central angles.

Have students draw two circles of different sizes. Have them draw a central angle of 75 degrees in each circle and then calculate the length of each arc intercepted by the 75 degree angles.
Lead a discussion as to how arc measure and arc length differ and under what conditions two arcs can have the same measure, but different lengths.

**Activity 6: Graph It! (GLEs: 13, 22)**

Have students work in groups to conduct surveys about favorite TV shows, foods, colors, etc. Each group should have results from different surveys which were conducted prior to this activity. Allow groups of students to create a circle graph to represent their data. Make sure that students use protractors to calculate the correct angle measures based on their data. Provide the students with the definition of the term *sector*. Once groups create their circle graphs, have them swap graphs and check each other’s work.

Provide students data sets and circle graphs that are already drawn with some of the graphs constructed incorrectly. Have students discuss the data that is presented to them in the graph and compare to the data sets provided. Using their knowledge of central angles, instruct students to determine if the circle graph has been constructed correctly. If any graph is incorrectly constructed, indicate that students should develop a new graph based on the given data. Instruct students to determine what each of the sectors represents. For instance, if one of the circle diagrams is a survey about favorite television programs and 24% of the 250 people surveyed like *The Frugal Millionaire*, have students determine the number of people who like that show.

**Activity 7: Major Versus Minor (GLE: 13)**

Provide students with data that can be put into a circle graph. Have students determine the kind of arc associated with each category of data (e.g., major, minor, semicircle). As an alternative, ask students to determine the type of arc associated with various times of the day displayed on an analog clock.

**Activity 8: Geometric Probability (GLE: 21)**

Have students practice finding the area of a sector using the formula $A = \frac{N}{360} \pi r^2$, where $N$ is the measure of a central angle. Give students circles (spinners) divided into unequal sectors. Have students find the probability of spinning and landing on a certain sector. Allow students to play simulated games to see if there is a way to always win the game if there are points allotted to certain sectors.

**Activity 9: Arcs and Chords (GLE: 13)**

Have students use a geometry software program (or compass and straightedge) to inscribe a variety of polygons in circles. Next, have students determine the measure and length of
each arc of the circle subtended by a chord. For example, inscribe a stop sign in a circle and then determine the two measures of each of the 8 arcs.

Provide student with an investigation guide to examine the relationship between a chord and its arc when a diameter is perpendicular to the chord, and the relationship between two chords that are equidistant from the center. Patty paper or the use of a drawing program such as Geometer’s Sketchpad® is appropriate for this investigation.

**Activity 10: Finding the Center (GLE: 13)**

Pose the problem of finding the center of a circular picnic table in order to cut a hole for an umbrella. Challenge students to use their knowledge of chords, lines perpendicular to a chord at its midpoint, and the intersection of these lines to find the center of the circle.

**Activity 11: Inscribed angles (GLEs: 13, 17, 19)**

Have students draw a circle and then inscribe a regular hexagon in the circle. Student should label the hexagon ABCDEF and the center of the circle P. Draw radii PA, PF, and PB. Label the following angles: ∠FPA as ∠1, ∠APB as ∠2, ∠FAP as ∠3, and ∠BAP as ∠4. Have students measure each numbered angle with a protractor and then find m∠FA and m∠AB and explain their reasoning. Next, instruct students to find m∠FAB and m∠BDF and make a conjecture about the relationship between m∠FAB and m∠BDF. Have students prove this conjecture and consider all three cases: the center of the circle lies on the side of the angle, the center of the circle is in the interior of the angle, and the center of the circle is in the exterior of the angle. Ask students to investigate the measure of an angle inscribed in a semicircle, and the measures of the angles in a quadrilateral inscribed in a circle.

**Activity 12: Tangents and Secants (GLE: 13)**

Provide students with diagrams that illustrate the properties of angles formed by tangents and secants. Listed are the three theorems to be discussed:

1. If a tangent and a secant intersect at a point on a circle, then the measure of each angle formed is half of the measure of its intercepted arc.
2. If two secants intersect in the interior of a circle, then the measure of each angle is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.
3. If a tangent and a secant, two tangents, or two secants intersect in the exterior of a circle, then the measure of the angle formed is half the difference of the measures of the intercepted arcs.
The first theorem is based solely on the inscribed angles discussed in Activity 11. To help illustrate the second theorem, have students use a compass to draw a circle of any radius and choose any point on the interior of the circle (not the center) and label it V. Then, using a straightedge, have students draw two secants that intersect at V. Label one secant \( \overline{WY} \) and the other secant as \( \overline{XZ} \). Label \( \angle ZVY \) as \( \angle 1 \). Next have them draw \( \overline{XY} \) and label \( \angle VYX \) as \( \angle 2 \) and \( \angle VXY \) as \( \angle 3 \). Students should see that \( \angle 2 \) and \( \angle 3 \) are inscribed angles so \( m\angle 2 = \frac{1}{2}m\widehat{WX} \) and \( m\angle 3 = \frac{1}{2}m\widehat{ZY} \). Lead the students in a discussion about why \( m\angle 1 = m\angle 2 + m\angle 3 \) and help them write an equation based on that understanding. Students should be able to understand the second theorem with this illustration.

The teacher will provide students with information about geostationary satellites and their orbits. Geostationary satellites move in a circular orbit about 26,000 miles above the Earth’s center. For a satellite whose orbit is directly over the equator, have students determine the measure of the arc along the equator that is “visible” to the satellite when given the measure of the exterior angle formed by the two tangent lines drawn from the satellite to the earth. For a satellite that is 26,000 miles from Earth’s center, the angle formed by two tangent lines measures approximately 17.7°. Ask students to apply the third theorem to find the measure of the required arc. Next, provide students with an arc measure along Earth’s equator that is less than \( \frac{1}{2} \) of Earth’s circumference (so that they have two secant lines intersecting at the satellite’s location). Have students determine the measure of an angle of view of the satellite.

Activity 13: Intersecting Chords and Secants (GLE: 13)

Have students use a drawing program such as The Geometer’s Sketchpad® to construct two intersecting chords, two intersecting secants, or a tangent and secant for a circle. These three cases are the basis of the Power of a Point Theorem. When point P lies inside the circle, the theorem is called the Intersecting Chords Theorem; when point P lies outside the circle, the theorem is called the Intersecting Secants Theorem.

Students should first prove that there are two similar triangles created if additional segments are added. \( \triangle APB \sim \triangle CPD \) in each case in the diagram provided on the next page. Once the similarity of the triangles is established, then students should be able to write the proportion \( \frac{AP}{CP} = \frac{BP}{DP} \), which is equivalent to the statement of the theorem:

\[ AP \cdot DP = BP \cdot CP. \]

Note that in the case of the tangent, points A and D coincide and are the same point.
Activity 14: Surface Area of a Sphere (GLE: 7)

Here is a concrete way to show students why the surface area of a sphere is derived from the formula $4\pi r^2$.

Students should already understand that the surface area of an object can be represented by how much wrapping paper it would take to cover it. Ask them to picture a sphere (a balloon or ball) and a piece of paper that is cut as wide as its diameter and as long as its circumference.

If you wrap the ball with the paper, you see that it would cover the entire sphere if it weren’t for all the overlaps (which would fit into the gaps if you cut them out). If possible, provide each student with a small sphere such as a tennis ball, softball, or golf ball. Have each student cut a rectangle from wrapping paper that will match the specifications listed for his/her ball and then test the concept.

Lead students through the algebraic development of the formula for the surface area of a sphere using this model as a starting point. The formula for the surface area of the paper is

$$\text{Length} \times \text{width} = \text{circumference} \times \text{diameter}$$
This is easily understood by looking at the picture above. Now substitute formulas we know:

\[ C = 2\pi r \text{ and } d = 2r \]
\[ C \times d = 4\pi r^2 = \text{surface area} \]

Provide practical applications problems which require the use of the lateral surface area of a sphere for students to work.

**Activity 15: Surface Area and Volume of Spheres (GLEs: 7, 17, 19)**

Spherical balls are used in many sports (e.g., golf, soccer, baseball, basketball). Have students research the various dimensions for selected balls. Students will then create a circle on paper that represents a great circle associated with the sphere (ball). Using the circle pattern, have students cut the circle into fractional sectors, each of which represents \( \frac{1}{8} \) of the circle. Next, have students “cover” half of the sphere (ball) with these sectors. Students will then make a conjecture about the surface area formula for a sphere. To conceptualize the volume formula, have students use centimeter or inch cubes and create a large cube that approximates the size of the ball.

**Activity 16: Surface Area and Volume of a Sphere (GLE: 7)**

Provide students with several types of balls (e.g., baseball, golf ball, basketball, soccer ball, tennis ball, etc.). Working in groups, students will determine the surface areas and volumes for each type of ball by first making appropriate measurements and then using those measurements in the correct formula.

**Sample Assessments**

**General Assessments**

- The student will complete journal entries for this unit. Journal topics could include:
  - Explain why the formula for the surface area of a sphere is \( 4\pi r^2 \) based on the activity performed in class.
  - Explain the differences between the secant of a circle and the tangent of a circle.
  - \( \triangle ABC \) is inscribed in a circle so that \( BC \) is the diameter. What type of triangle is \( \triangle ABC \)? Explain your reasoning.
- The student will find pictures in magazines or newspapers of diagrams that show tangents, secants, and chords. The student will explain what the picture
is and why it represents the term they are defining. These pictures could be included in a portfolio.

- The student will construct various circles with different areas. The student will construct specified inscribed angles, arcs of given measures, secants, and tangents.

**Activity-Specific Assessments**

- **Activity 3:** The student will create a dartboard game using area properties of circles or other figures. The student will determine the probabilities involved with the game and then play the games to determine the experimental probability.

- **Activity 5:** The student will find examples of circle graphs in magazines or newspapers. He/she will write a paragraph that describes the information presented in the graph and find the measures of the central angles based on the information provided in the graph. He/she will determine if the graph has been constructed correctly.

- **Activity 15:** The student will:
  1. Find the volume of a tennis ball can. The student will either take measurements from an actual can or the information will be provided in a diagram by the teacher.
  2. Assuming that the tennis balls are tightly packed, find the total volume of the three tennis balls.
  3. Determine the percentage of the volume of the can taken up by the tennis balls.
  4. Determine the volume of sand needed if sand is poured to fill the remaining air space in the can.
  5. Calculate the percentage of the can’s volume taken up by the sand if one of the tennis balls is removed and sand is poured in to replace it.
Geometry
Unit 8: Transformations

Time Frame: Approximately three weeks

Unit Description

This unit provides a deeper mathematical understanding and justifications for transformations that students have seen in previous grades. The focus is providing justifications for the congruence and similarity relationships associated with translations, reflections, rotations, and dilations (centered at the origin).

Student Understandings

Students can determine what transformations have been performed on a figure and can determine a composition of transformations that can be performed to mimic other transformations like rotations. They are also able to find new coordinates for transformations without actually performing the indicated transformation.

Guiding Questions

1. Can students find transformations and mappings that relate one congruent figure in the plane to another?
2. Can students provide an argument for the preservation of measures of figures under reflections, translations, and rotations?
3. Can students find the dilation (magnification), centered at the origin, of a specified figure in the plane and relate it to a similarity mapping?
4. Can students perform a composition of transformations and explain its relationship to single transformations or other compositions that produce the same image?
5. Can students solve 2- and 3-dimensional problems using transformations?
Unit 8 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.</td>
<td>Develop and apply coordinate rules for translations and reflections of geometric figures (G-3-H)</td>
</tr>
<tr>
<td>15.</td>
<td>Draw or use other methods, including technology, to illustrate dilations of geometric figures (G-3-H)</td>
</tr>
<tr>
<td>17.</td>
<td>Compare and contrast inductive and deductive reasoning approaches to justify conjectures and solve problems (G-4-H) (G-6-H)</td>
</tr>
<tr>
<td>19.</td>
<td>Develop formal and informal proofs (e.g., Pythagorean theorem, flow charts, paragraphs) (G-6-H)</td>
</tr>
</tbody>
</table>

Sample Activities

Activity 1: Understanding Congruence, Similarity, and Symmetry Using Transformations and Interactive Figures: Visualizing Transformations (Using Technology) (GLEs: 14, 15)

Use the following website to provide an investigation of various kinds of transformations, including dilations: http://www.utc.edu/~cpmawata/transformations/translations/. In this activity, have students move an object and view the object and its image as a reflection occurs. The line of reflection can be shown or hidden.

The website, http://standards.nctm.org/document/eexamples/chap6/6.4/, allows students to visualize transformations and compositions of transformations while working interactively with various geometric figures. Students explore the effects of applying reflections, translations, and rotations to any one of three shapes. Ask questions, such as “How does the original shape compare to the shape after the transformation?” and “What is the effect of the transformation on the side lengths and angle measures of the original shape?”

Activity 2: A Basic Look at Transformations (GLE: 14)

Provide students with a sheet of graph paper with the four quadrants marked. In Quadrant 1, have students construct a polygon by providing a set of coordinates for the vertices. Next, instruct students to perform various translations and reflections of the shape. Have students develop a series of translations or reflections that combine to produce the original shape in its original location. After each transformation, have students determine the vertices of the transformed polygon. For example, if the polygon is reflected over the $x$-axis, then each vertex will have the same $x$-coordinate as the original, but the $y$-coordinates will be the opposites of the original.
Activity 3: Understanding Reflections (GLEs: 14, 17)

Give students pictures of various reflections using the x-axis, y-axis, the origin, and the line $y = x$. Have students develop a chart that names the type of reflection, the change of the original to the image, a statement about how to find the coordinates of the image, and a numerical example in the coordinate plane. For example, a chart might look like the following:

<table>
<thead>
<tr>
<th>Reflection</th>
<th>Over x-axis</th>
<th>Over y-axis</th>
<th>Around Origin</th>
<th>Over $y = x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original to image</td>
<td>(a, b) $\rightarrow$ (a, -b)</td>
<td>(a, b) $\rightarrow$ (-a, b)</td>
<td>(a, b) $\rightarrow$ (-a, -b)</td>
<td>(a, b) $\rightarrow$ (b, a)</td>
</tr>
<tr>
<td>Statement</td>
<td>Multiply y-coordinate by –1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Example</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Provide students with multiple examples for various different figures with different properties to complete this chart. Next, provide students with coordinates of a figure and have them determine the coordinates of the image points for a given reflection without graphing any of the points. Allow students to use the chart to determine what the coordinates of the image points should be. After they have identified the new coordinates for each reflection, have them graph their original images and the reflected images to check their work.

Activity 4: Understanding Rotations (GLEs: 14, 17)

*Teacher Note: While GLE 14 does not refer to rotations, rotations are tested on the GEE21. This activity will also serve as a precursor to the GLEs for grades 11 and 12.*

Give students a pre-image on the coordinate plane with the vertices labeled. Have students make a chart like the one below and rotate the pre-image $90^\circ$, $180^\circ$, $270^\circ$, and $360^\circ$ using the origin as the center of rotation. They should record the new coordinates each time and then graph the new images.

<table>
<thead>
<tr>
<th>Original</th>
<th>$90^\circ$</th>
<th>$180^\circ$</th>
<th>$270^\circ$</th>
<th>$360^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a,b)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c,d)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(e,f)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(g,h)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x,y)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

After drawing the new figures, have students analyze the coordinates to determine a pattern of changes in coordinates. Provide opportunities to practice performing rotations both with and without this chart on the coordinate plane.
Activity 5: Slide It! (GLE: 14)

Provide students with diagrams showing various polygons on a coordinate plane. Give specific instructions about the directions in which to move the figure (3 units up and 4 units right, 2 units down and 4 units left, etc). Have students record the original and new coordinates of the vertices and analyze the results. Lead a discussion to summarize how the x-coordinate is affected when a point is translated left/right and how it is affected if the point is moved up/down. Have similar discussion about the change in the y-coordinate when the point is moved left/right and up/down. Assuming that \(a\) represents a horizontal translation and \(b\) represents a vertical translation, the effect of the translation is point \((x, y) \rightarrow (x+a, y+b)\).

Provide the opportunity for students to draw a translation that moves in the same direction as a given vector. The vector may be already drawn on the coordinate plane so that students can identify the number of units to move. Again, they should analyze their ordered pairs and determine if the pattern.

Activity 6: Magnify It! (GLE: 15)

Have students work in groups of two to develop a specific dilation of a figure that has been graphed in the plane. First, have them create a dilation that is 1.5 times the size of the original figure. Next, instruct students to create a dilation that is .75 times the size of the original figure. Be sure to instruct students to specify the coordinates of the dilated figures. If available, use *The Geometer’s Sketchpad*® or other drawing program to perform dilations. Have students make a statement about the similarity of the original figure and its dilation and determine the attributes of the original figure that remain unchanged after the dilation is performed.

Activity 7: Make a Conjecture and Prove It! (GLEs: 14, 15, 17, 19)

Using a geometry software package such as *The Geometer’s Sketchpad*®, have students create several translations, reflections, rotations, or dilations and combinations of these and then examine the properties of the transformed figures compared to original figures. Using these inspections, have students make conjectures about the effects of these transformations including conjectures concerning congruence and similarity. Instruct students to prove their conjectures.

For example, are any of the combinations of transformations the same as a single type of transformation? Are transformations commutative, that is, can you change the order of two transformations and get the same result? Allow students to use inductive and deductive reasoning while comparing conjectures and accompanying proofs. To enhance student understanding, have students begin by reflecting their figures over two parallel lines and compare this with a translation. Next, have students reflect their figures over a pair of intersecting lines (e.g., the x- and y-axes) and compare this with a rotation. Be sure to have students perform this activity through several iterations. Each iteration should focus on a
specific set of combined transformations. As an alternative, the teacher could provide students with a pre-image and an image in the coordinate plane with vertices labeled and require students to determine the transformation or set of transformations that produced the image.

Sample Assessments

General Assessments

- The teacher will provide the student with a polygon in the coordinate plane and instruct the student to perform various transformations on it.
- The student will investigate the transformations that are used in a board game, such as “checkers” or “chess.”
- The student will create a portfolio containing samples of work from the activities. Portfolio entries will include copies of the transformations performed in class with explanations about the procedures used to complete the transformation.
- The student will write journal entries that are graded. Topics might include:
  - Which transformations produce congruent figures? Why are these figures congruent to their originals?
  - How would you define the words reflection, rotation, translation, and dilation? Are your definitions of these words different from your previous definitions since the completion of this unit? (This journal would be posed twice: the first time, only the first question would be asked, and would be posed before the unit; the second time would be after the unit and both questions would be asked.)
- The student will create a “Transformation Album.” He/she will create a figure and perform at least one of each of the different transformations on the figure in the coordinate plane. The teacher will assess the work based on the accuracy of the transformations.

Activity-Specific Assessments

- **Activity 3**: The teacher will ask the student to demonstrate his/her ability to do the following:
  1. Given \(A(-2,3), B(-5,7),\) and \(C(-1, 10)\), graph triangle \(ABC\).
  2. Reflect triangle \(ABC\) over the \(x\)-axis. Label the image as \(A'B'C'\).
  3. Find the area of triangle \(A'B'C'\).
  4. Explain how the area of triangle \(A'B'C'\) compares to the area of triangle \(ABC\).

- **Activity 4**: The teacher will provide the student with the coordinates of an image that has already been rotated \(90^\circ\), \(180^\circ\), or \(270^\circ\). The student will find the coordinates of the pre-image using either the coordinate plane or the chart developed in this activity.
• **Activity 8**: The student will design a tessellation using glide reflections, rotations, or translations. If materials are available, the student will transfer the design to a cloth square in order to make a class quilt. This can be done using special crayons which can be found in some school and/or art supply stores.